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FAST SECOND ORDER ORIGINAL PRONY'S METHOD FOR EMBEDDED MEASURING SYSTEMS

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Abstract

The paper presents a method of adaptation of the original second order Prony's method for applications in low-cost digital measurement systems with low computing performance. The presented method can be used in measuring systems where it is important to obtain in real time the values of amplitude, frequency, initial phase and damping coefficient of a single sinusoidal component of an analysed signal. The paper presents optimized, in terms of the number of mathematical operations, implementation of the method in selected embedded devices as well as the calculation times of the method for each platform.

Keywords: Prony's method, signal processing, harmonics, measurements.

1. Introduction

Analysis of a digital multi-frequency signal to estimate the basic parameters of its components is a very broad subject mainly related to Fourier transforms [1−7]. One section of this area includes the analysis dedicated to estimation of a single component with the greatest possible accuracy and short calculation time. This area of application includes some modifications of the Fourier analysis [8−9] and other methods [10], of which the Prony's methods gain greater and greater practical significance [11−16].

The Prony's methods are characterized by the properties of precise estimation of the parameters of an analysed signal. They generate new measurement possibilities by identifying real frequencies of the analysed signal components, and by extending the signal model with information about the damping coefficients of the components [17–20]. These methods enable the use of short windows of analysis, which is valuable in the study of fast variable phenomena. They also specify the parameters of components of slowly variable signals, with incomplete periods in the analysed analysis window. Nevertheless, when analysing multiple components, they are computationally complex methods that require inversion of large matrixes, and calculation of roots of high-order polynomials. These methods may also involve problems with the numerical stability of solutions.

The great versatility of Prony's methods makes them an alternative to Fourier transform-based methods, enabling to measure a wider range of signals in variable measurement conditions (analysis window duration, sampling frequency) not available with other methods.

The paper presents a method of modifying the calculations required in the algorithm of original version of Prony's method of the second order in such a way as to obtain maximum simplification. The proposed modification is based on fundamental mathematical operations without involving complex numbers and operations requiring iterative calculations. This enables a significant reduction of the calculation time of Prony's method even for low-performance embedded devices. In the paper the accuracy aspect of calculations of Prony's

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method is deliberately omitted, as it is the subject of separate publications [17–20]. The implementation of the method for a specific application is also not presented so as not to narrow down the group of potential recipients of the proposed solution.

The paper consists of 4 Sections. In Section 1 an introduction is included. In Sector 2 there are described the original Prony's method and its modifications to simplify the calculations of the presented algorithm. Section 3 shows the implementation of the method in selected embedded devices and the measurement results of algorithm execution time. Sector 4 contains a summary.

2. Description of adaptation of original Prony's method for embedded applications

The original Prony's method can be presented essentially as two calculation stages. In the first stage, frequency and damping coefficients of the complex exponents modelling the analysed signal are determined. In the second stage, amplitudes and initial phases of the components are calculated based on the parameter values determined in the previous stage [17, 21].

2.1. Determination of frequency and damping coefficients of components

In the first stage of the original Prony's method calculations a Toeplitz matrix is created on the basis of samples $x_1 \ldots x_{2p}$ of analysed signal, (left side of (1)). The following equation is based on this matrix:

$$\begin{bmatrix} x_p & x_{p-1} & \cdots & x_1 \\ x_{p+1} & x_p & \cdots & x_2 \\ \vdots & \vdots & \cdots & \vdots \\ x_{2p-1} & x_{2p-2} & \cdots & x_p \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix} = -\begin{bmatrix} x_{p+1} \\ x_{p+2} \\ \vdots \\ x_{2p} \end{bmatrix}, \tag{1}$$

where p is a size of Prony's model, and the vector $A_1 \ldots A_p$ is a set of certain coefficients, which will be described later in the argument. For the adopted order of Prony's model $p = 2$, the solution of (1) can be represented by a relation:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = -\left(\begin{bmatrix} x_p & x_{p+1} \\ x_{p-1} & x_p \end{bmatrix} \cdot \begin{bmatrix} x_p & x_{p-1} \\ x_{p+1} & x_p \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} x_p & x_{p+1} \\ x_{p-1} & x_p \end{bmatrix} \cdot \begin{bmatrix} x_{p+1} \\ x_{p+2} \end{bmatrix}. \tag{2}$$

To increase the transparency, let, $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$. By making further transformations related to the calculation of the inverse matrix we obtain:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = -\frac{1}{r} \begin{bmatrix} a^2 + b^2 & -ab - bc \\ -ab - bc & b^2 + c^2 \end{bmatrix} \cdot \begin{bmatrix} b & c \\ a & b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix}, \tag{3}$$

where:

$$r = \left(a^2 + b^2\right)\left(b^2 + c^2\right) - \left(ab + bc\right)^2 \tag{4}$$

is a determinant of the inverted matrix. Estimation of component parameters of Prony's model takes place for $r \neq 0$. Finally, it can be written as:

$$A_1 = -\frac{bc\left(a^2 + b^2\right) - ac\left(ab + bc\right) + cd\left(a^2 + b^2\right) - bd\left(ab + bc\right)}{r}, \tag{5}$$

$$A_2 = -\frac{ac\left(b^2 + c^2\right) - bc\left(ab + bc\right) + bd\left(b^2 + c^2\right) - cd\left(ab + bc\right)}{r}. \tag{6}$$

In this way coefficients of a polynomial of the general form are determined:

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$$\phi(z)=\sum_{m=0}^{p} A_m z^{p-m} \tag{7}$$

for which the next step is to determine its complex roots z_k:

$$\phi(z)=\prod_{k=1}^{p}(z-z_k). \tag{8}$$

It is assumed that $A_0 = 1$ [21]. For the considered case $p = 2$, a square polynomial of the general form is obtained:

$$\phi(z)=A_0 z^2 + A_1 z + A_2 , \tag{9}$$

where, knowing the coefficients A_0, A_1, A_2 , the zero of the function can be determined using the commonly known Vieta's formulas. For the polynomial (9) we can first write:

$$\Delta = A_1^2 - 4 A_0 A_2 . \tag{10}$$

The estimation of the sinusoidal components of Prony's model is obtained for complex roots, *i.e.* for $\Delta < 0$. For the Prony's model with $p = 2$, we obtain conjugate roots describing sinusoidal components damped: one with a positive frequency and the other with identical amplitude, initial phase and damping but with a negative frequency. By further transformation of Vieta's formulas, we obtain the solution:

$$\mathrm{Re}\{z_1\} = \mathrm{Re}\{z_2\} = -\frac{A_1}{2A_0}, \tag{11}$$

$$\mathrm{Im}\{z_1\} = -\mathrm{Im}\{z_2\} = \frac{\sqrt{-\Delta}}{2A_0}. \tag{12}$$

The roots in the first stage of Prony's method are complex but for further embedded applications the operations can be performed for real data types, as – based on the real part $\mathrm{Re}\{z_1\}$ and imaginary part $\mathrm{Im}\{z_1\}$ of selected individual root – the frequencies f_1 and f_2 of the components can be calculated according to the following relation:

$$|z_1| = |z_2| = \sqrt{\frac{A_1^2 - \Delta}{4 A_0^2}}, \tag{13}$$

$$f_1 = -f_2 = \frac{1}{2\pi T}\arcsin\left(\frac{\mathrm{Im}\{z_1\}}{|z_1|}\right), \tag{14}$$

where T is a sampling period of the analysed signal, whereby, if the estimated component is not damped, then $|z_1| = |z_2| = 1$. The damping coefficients α_1 and α_2 can be determined from the relation:

$$\alpha_1 = \alpha_2 = \frac{1}{T}\ln(|z_1|). \tag{15}$$

Finally, by making simple transformations, we can determine the frequency and damping coefficients of an estimated component using a simple C code or Matlab:

```
a2=a*a; b2=b*b; c2=c*c; ab=a*b;
bc=b*c; cd=c*d; ac=a*c; bd=b*d;
a2_b2=a2+b2; b2_c2=b2+c2; ab_bc=ab+bc;
r=a2_b2*b2_c2-ab_bc*ab_bc;
A1=-(bc*a2_b2-ac*ab_bc+cd*a2_b2-bd*ab_bc)/r;
A2=-(ac*b2_c2-bc*ab_bc+bd*b2_c2-cd*ab_bc)/r;
del=A1*A1-4*A2; rez1=-A1/2; imz1=sqrt(-del)/2;
absz1=sqrt(rez1*rez1+imz1*imz1);
F=asin(imz1/absz1)/(2*pi*T);
AL=log(absz1)/T;
```

For calculation of the first stage of Prony's model $p = 2$ in the original version, it is required to perform in total: 24 multiplications, 12 additions, 5 divisions, and additionally 2 root extractions and 1 arcsin operation. All operations are performed on real data types.

2.2. Determination of initial stages and amplitudes

In the second stage of the original Prony's method, the first operation is to determine a Vandermonde matrix (16):

$$V = \begin{bmatrix} z_1^0 & z_2^0 & \cdots & z_p^0 \\ z_1^1 & z_2^1 & \cdots & z_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ z_1^{p-1} & z_2^{p-1} & \cdots & z_p^{p-1} \end{bmatrix} \tag{16}$$

from complex roots z_k and solving the (17):

$$h = \left(V^{\mathrm{T}} V\right)^{-1} V^{\mathrm{T}} x, \tag{17}$$

where: $h = [h_1,\ldots,h_p]^{\mathrm{T}}$, and $x=[x_1,\ldots,x_p]^{\mathrm{T}}$.

The determined vector h is used in the next step to calculate amplitudes *amp* and initial stages φ of the components of Prony's model, according to the relation [21]:

$$amp = |h|, \tag{18}$$

$$\varphi = \arcsin\left(\frac{\mathrm{Im}\{h\}}{|h|}\right). \tag{19}$$

When using a model with size $p = 2$, the number of mathematical operations to perform is relatively small. However, multiplications and additions on complex numbers are required, which greatly increases the number of operations based on real data types. For example, 1 complex multiplication translates into 4 multiplications and 2 additions of real numbers, and one complex addition translates into 2 additions of real numbers. Therefore, the direct adaptation of the second stage of Prony's method for embedded applications is not favourable in terms of the number of mathematical operations.

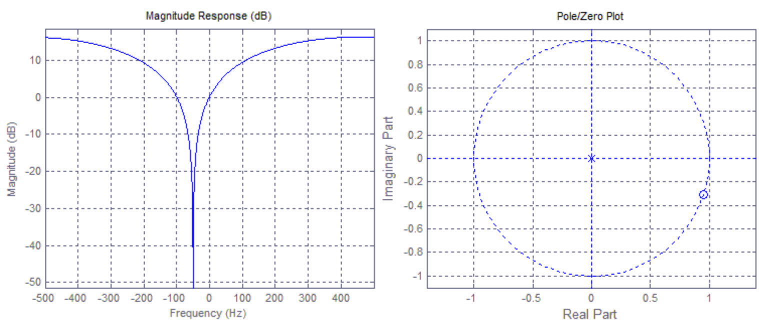

Fig. 1. The gain characteristics and positions of zeros of an FIR filter with coefficients C_{row1} calculated for a sinusoidal signal $f = 50$ Hz, $\alpha = 0$ with $T = 1$ ms.

The key to reducing the computational complexity of this stage lies in the observation that the term $(V^{\mathrm{T}} V)^{-1} V^{\mathrm{T}}$ in the relation (17) is a certain constant for the frequencies and coefficients of damping with a p x p matrix (C). The last step in calculation of the relation (17) is therefore

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the product of the matrix C and the vector of analysed signal x. For $p = 2$, this operation can be replaced by two operations of vector multiplication:

$$h_1 = C_{row1}x \text{ and } h_2 = C_{row2}x, \tag{20}$$

whereby, the solution $h_1 = \text{conj}(h_2)$ is obtained in the same way as $z_1 = \text{conj}(z_2)$. Therefore, it is enough to determine the parameters of a single element of vector h. Note that the operations from (20) are filter operations with FIR type filters with coefficients: C_{row1} for h_1 of signal x and C_{row2} for h_2 of signal x. However, the complex calculation of single FIR filter coefficients is still required, as described above. Further observations can be made by observing the performance characteristics of FIR filter with coefficients C_{row1} – Fig. 1.

It turns out that for the original Prony's method with $p = 2$, if $\alpha = 0$, we can write equality roots(C_{row1}) = z_1 and roots(C_{row2}) = z_2, but for $\alpha \neq 0$ angle(roots(C_{row1})) = angle(z_1) and angle(roots(C_{row2})) = angle(z_2). Based on these observations, regarding the assumptions of the model order, the second stage of Prony's method can be simplified. This simplification will enable to determine the initial stages of the components.

In the first stage, on the basis of a previously determined single zero position z_1 we determine the coefficients of a certain polynomial $\Psi(z)$ of first degree; let:

$$\Psi(z) = B_1 z + B_0. \tag{21}$$

Having the root z_1 of this polynomial, based on the relation describing the zero of the equation of the straight line:

$$z_1 = \frac{-B_0}{B_1} \tag{22}$$

we can write down:

$$\Psi(z) = B_1 z - B_1 z_1. \tag{23}$$

In the next step, we can create a vector with the coefficients of the searched FIR filter:

$$C'_{row1} = [-B_1 z_1 \quad B_1]. \tag{24}$$

However, the parameter B_1 is still unknown. Exact knowledge of this parameter would enable estimation of the amplitude and phase of the desired components. It turns out, however, that to obtain the correct estimation of only the initial phase, it is enough to accept the value from the example shown, *i.e.*: 0–3.236 for $B1$ or another value with a zero real part and a negative imaginary part. For further transformations for simplicity, it was assumed that $B_1 = -i$. In this way, substituting $B_1 = -i$ in the formula (24) and next in the formula (20), and substituting $x_1 = a$ and $x_2 = b$ after transformations, it can be written down as a simple relation to the real and imaginary parts of the parameter h_1:

$$\text{Re}\{h_1\} = -\text{Im}\{z_1\} \cdot a, \tag{25}$$

$$\text{Im}\{h_1\} = \text{Re}\{z_1\} \cdot a - b. \tag{26}$$

By substituting the determined parameter in (19), a valid initial phase of a single component can be determined. The number of operations required for this purpose is small and is based on real data types. However, another solution should be used to determine the correct amplitude of Prony's model. The easiest way is to use the relation:

$$x_1 = amp \cdot \cos(2\pi f t_1 + \varphi). \tag{27}$$

If we make calculations for time $t_1 = 0$, with the substitution $x_1 = a$ we will obtain:

$$amp = \frac{a}{\cos(\varphi)}. \tag{28}$$

Ultimately, the calculation of the initial stage and the amplitude of Prony's model can be reduced to just a few lines of code in C or Matlab:

```
reh1=-imz1*a;
imh1=rez1*a-b;
absh1=sqrt(reh1*reh1+imh1*imh1);
FI=asin(imh1/absh1);
AMP=a/cos(FI).
```

The second stage of Prony's method requires a total of 4 multiplications, 2 additions, 2 divisions, and 1 root extraction operation, 1 arcsin and 1 cosine operation. All operations are performed on real data types.

3. Implementation of method in embedded device

The C-code method described in Section 2 was implemented in embedded devices with microprocessors of different architectures. These were the following microprocessors:

1) NUC140VE3CN in module Nuvoton Nu-LB-NUC140 [22];
2) LM4F120H5QR in module Stellaris LM4F120 LaunchPad Evaluation Kit [23];
3) TMS320C28027 in module C2000 Piccolo LaunchPad LAUNCHXL-F28027 [24];
4) MSP430G2553 in module MSP-EXP430G2 TI LaunchPad [25].

The execution times of the method were analysed for selected hardware platforms. The results are shown in Table 1. All calculations, except for the selected case, were made for 64-bit double numbers.

Table 1. Comparison of computation times for different processor systems.

Processor	clock [MHz]	computation time [μs]
LM4F120H5QR	80	215.1
TMS320C28027	60	275.4
NUC140VE3CN	50	337.1
MSP430G2553 (32bit)	16	1161.1
MSP430G2553 (64bit)	16	3733.0

The execution times of individual instruction groups were also analysed in relation to the execution time of the entire algorithm. The analysis was performed for the Nuvoton Nu-LB-NUC140 platform and the results are presented in Table 2.

Table 2. A summary of the numbers of operations and their execution times by the original Prony's method with $p = 2$, for Nuvoton Nu-LB-NUC140, with $f_{CLK} = 50$ MHz.

operation	number of operations I stage	number of operations II stage	operations in total	execution time 1 instruction [μs]	execution time instructions [μs]	instruction percentage
*	24	4	28	4.2	117.6	35%
+	12	2	14	2.3	32.2	10%
/	5	2	7	7.7	53.9	16%
$\sqrt{\ }$	2	1	3	11.7	35.1	10%
arcsin	1	1	2	25.0	50.0	15%
cos	0	1	1	25.0	25.0	7%
ln	1	0	1	23.3	23.3	7%
total:	**45**	**11**	**56**		**337.1**	**100%**

4. Conclusions

A modification of Prony's method presented in the paper enables simple implementation of the original Prony's second-order method in embedded devices with a low computing power. The optimization of the algorithm enabled to use short algorithms in a wide range of measurement devices that perform measurement of a single sine component of an analysed signal and in security devices that have a short response time to specific events. Because the order of the model is limited in the algorithm to one real component for practical applications in which the analysed signal contains different distortions, the best results of the method can be achieved using an additional bandpass filter in the algorithm prior to the analysis by Prony's method [26]. The basic parameters of the bandpass filter, such as bandwidth, waving, and stopband damping should be chosen in accordance with a specific implementation. Examples of implementation of the method for specific measurement applications along with the selection of a bandpass filter will be the subject of further publications.

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