

# Stability and controllability of switched systems

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**Abstract.** The study of properties of switched and hybrid systems gives rise to a number of interesting and challenging mathematical problems. This paper aims to briefly survey recent results on stability and controllability of switched linear systems. First, the stability analysis for switched systems is reviewed. We focus on the stability analysis for switched linear systems under arbitrary switching, and we highlight necessary and sufficient conditions for asymptotic stability. After that, we review the controllability results.

**Key words:** hybrid systems, stability, Lyapunov function, characteristic exponents, controllability.

## 1. Introduction

Hybrid systems which are capable of exhibiting simultaneously several kinds of dynamic behaviour in different parts of the system (e.g., continuous-time dynamics, discrete-time dynamics, jump phenomena, logic commands, and the like) are of a great current interest (see, e.g., [1–3]). Examples of such systems include the Multiple-Models, Switching and Tuning paradigm from adaptive control [4], Hybrid Control Systems [5], and a plethora of techniques that arise in Event Driven Systems. Also typical examples of such systems of varying degrees of complexity include computer disk drives [6], transmission and stepper motors [7], constrained robotic systems [8], intelligent vehicles/highway systems [9], sampled-data systems [10], discrete event systems [11], and many other types of systems (refer, e.g., to the papers included in [2]).

Switched linear systems are hybrid systems that consist of several linear subsystems and a rule of switching among them. Switched linear systems provide a framework which bridges the linear systems and the complex and/or uncertain systems. On one hand, switching among linear systems may produce complex system behaviors such as chaos and multiple limit cycles. On the other hand, switched linear systems are relatively easy to handle as many powerful tools from linear and multilinear analysis are available to cope with these systems. Moreover, the study of switched linear systems provides additional insights into some long-standing and sophisticated problems, such as intelligent control, adaptive control and robust analysis.

A theoretical examination of switched linear systems is academically more challenging due to their rich, diverse and complex dynamics. Switching makes these systems much more complicated than standard-time invariant or even time-varying systems. Many more complicated behaviours/dynamics and fundamentally new properties, which standard systems do not have, have been demonstrated on switched linear systems. From the control system design point of view, switching brings an additional degree of freedom in

control system design. Switching laws, in addition to control laws, may be utilized to manipulate switched systems to achieve a better performance of a system. This can be treated as an added advantage for a control design to attain certain control purposes like stabilizability or controllability.

The objective of this article is to review the major progress that has been made on stability and controllability of switched linear systems over the past number of years, see also [12] and [13]. As a part of this process we attempt to outline the major outstanding issues that have yet to be resolved in the study of switched linear systems.

Controllability is one of the fundamental concepts in the mathematical control theory. This is a qualitative property of the dynamical control systems and is of particular importance in a control theory. The systematic study of controllability was started at the beginning of sixties in the last century when the theory of controllability based on the state space description for both, time-invariant and time-varying linear control systems, was introduced.

Roughly speaking, controllability means, that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. It should be mentioned that in the literature there are many different definitions of controllability, which strongly depend on a class of dynamical control systems on one hand, and on the other hand, on admissible controls [14–16].

Controllability problems for different types of dynamical systems require the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, matrix analysis and theory of ordinary and partial differential equations and theory of difference equations. In the paper we use mainly state-space models of dynamical systems, which provide a robust and universal method for studying controllability of various classes of systems [14–16].

Controllability plays an essential role in the development of the modern mathematical control theory. There are various important relationships between controllability, stability and

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stabilizability of linear both finite-dimensional and infinite-dimensional control systems. Controllability is also strongly related to the theory of realization and so-called minimal realization and canonical forms for linear time-invariant control systems such as the Kalman canonical form, the Jordan canonical form or the Luenberger canonical form [14].

It should be mentioned, that for many dynamical systems there exists a formal duality between the concepts of controllability and observability. Moreover, controllability is strongly connected with a minimum energy control problem for many classes of linear finite dimensional, infinite dimensional dynamical systems, and delayed systems both deterministic and stochastic [14–16].

Moreover, it is well known, that the controllability concept has many important applications not only in the control theory and systems theory, but also in such areas as industrial and chemical process control, reactor control, control of electric bulk power systems, aerospace engineering and recently in the quantum systems theory.

The last decades have seen a continually growing interest in the controllability theory of dynamical systems. This is clearly related to the wide variety of theoretical results and possible applications. Up to the present time the problem of controllability for continuous-time and discrete-time linear dynamical systems has been extensively investigated in many papers (see e.g. [14] for extensive list of references).

Similarly, there have been a lot of papers for controllability both continuous-time and discrete-time switched systems [17–23]. For the controllability analysis of switched linear control systems, a much more difficult situation arises since both the control input and the switching rule have been design variables to be determined. Thus, the interaction between them is very important from the controllability point of view. Moreover, it should be mentioned that for the switched linear discrete-time control system in a general case the controllable set is not a subspace but a countable union of subspaces. For the switched linear continuous-time control system, in a general case the controllable set is an uncountable union of subspaces.

## 2. Definitions and model descriptions

Consider a set of square  $n$ -by- $n$  matrices  $\Sigma = \{A_i : i \in I\}$ . Throughout this paper our primary concern shall be with the stability properties of the switched linear system. In a discrete-time case it has the following form

$$x(k+1) = A_{\sigma(k)}x(k), \quad (1)$$

where  $\sigma : \mathbb{N} \rightarrow I$ , and in continuous-time a switched linear system has the form

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad (2)$$

where  $\sigma : [0, \infty) \rightarrow I$  is a piecewise constant function. The function  $\sigma$  in both cases is called the switching signal. In the continuous-time case the points of discontinuity,  $t_1, t_2, \dots$ , of  $\sigma$  are known as the switching instances. We denote the set of switching signals by  $S(I)$ . A function  $x : [0, \infty) \rightarrow \mathbb{R}^n$  is

called a solution of (2) if it is continuous and piecewise continuously differentiable and  $\dot{x}(t) = A_{\sigma(t)}x(t)$  for all  $t$  except at the switching instances of  $\sigma$ . Considering problems of controllability and stabilizability our systems have the following forms

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad (3)$$

where  $\sigma : \mathbb{N} \rightarrow I$  and in continuous-time a switched linear system has the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad (4)$$

where  $\Sigma_C = \{B_i : i \in I\}$  is a set of  $n$ -by- $p$  matrices and  $u : [0, \infty) \rightarrow \mathbb{R}^p$  is a control. For a given switching function  $\sigma$  and in case of (3), (4) for given control  $u : [0, \infty) \rightarrow \mathbb{R}^p$  the solution of (2)–(4) with an initial condition  $x(0) = x_0 \in \mathbb{R}^n$  is denoted by  $x(k, \sigma, x_0)$  ( $x(t, \sigma, x_0)$ ,  $x(k, \sigma, x_0, u)$ ,  $x(t, \sigma, x_0, u)$ ).

## 3. Stability

For  $i \in I$  the time-invariant system

$$x(k+1) = A_i x(k), \quad \dot{x}(t) = A_i x(t) \quad (5)$$

will be called subsystem of (1), (2).

The stability issues of such switched systems include several interesting phenomena. For example, even when all the subsystems (5) are exponentially stable, (1), (2) may have divergent trajectories for certain switching signals  $\sigma$ ; see, e.g. [12, 24]. Another noticeable fact is that one may carefully switch between unstable subsystem to make (1) or (2) exponentially stable; see, e.g. [25]. As these examples suggest, the stability of switched systems depends not only upon the dynamics of each subsystems but also upon the properties of the switching signals. Therefore, the stability study of switched systems might be roughly divided into two kinds of problems [25]:

- (Q1) one is the stability analysis of switched systems under given sets of admissible switching signals (all switching signals or these obeying some constraints);
- (Q2) the other is the synthesis of stabilizing switching signals for a given collection of dynamical/control systems.

**Definition 1.** The system (1), (2) is called absolutely stable if there exist real constants  $M \geq 1$ ,  $\beta > 0$  such that

$$\|x(k, \sigma, x_0)\| \leq M e^{-\beta k} \|x_0\|$$

$$(\|x(t, \sigma, x_0)\| \leq M e^{-\beta t} \|x_0\|)$$

for all  $x_0 \in \mathbb{R}^n$ ,  $\sigma$  and  $k$  ( $t$ ).

**3.1. Common quadratic function approach.** The existence of a common quadratic Lyapunov function (CQLF) for all its subsystems assures absolute stability. However, the so-called quadratic stability is much stronger condition than absolute stability. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and has attracted a lot of research efforts due to its importance in practice. It is known that the conditions for the existence of a CQLF can be expressed as linear matrix inequalities.

Recall that  $V(x) = x^T P x$  is the quadratic Lyapunov function (QLF) for the system  $\dot{x} = Ax$  ( $x(k+1) = Ax(k)$ ) if  $P$  is symmetric and positive definite, and  $PA + A^T P$  ( $A^T P A - P$ ) is negative definite. Let  $\Sigma$  be a collection of  $n$ -by- $n$  Hurwitz (Schur) matrices, with associated stable subsystems (5). Then the function  $V(x) = x^T P x$  is a common quadratic Lyapunov function (CQLF) for these systems if  $V$  is a QLF for each individual system. Given a set of matrices  $\Sigma$ , the CQLF existence problem is to determine whether such a matrix  $P$  exists. A secondary question is to construct a CQLF when one is known to exist. It is a standard fact that time-invariant system  $\dot{x} = Ax$  ( $x(k+1) = Ax(k)$ ) has a QLF if and only if the matrix  $A$  is Hurwitz (Schur). This property is also equivalent to the exponential stability of the system  $A$ , so for a single time-invariant system there is no gap between the existence of a QLF and exponential stability. For a collection of Hurwitz (Schur) matrices the situation is more complicated in several aspects. Firstly, in general, CQLF existence is only a sufficient condition for the absolute stability of a switched linear system. Secondly, no correspondingly simple condition is known which can determine the existence of a CQLF for a family of time-invariant systems, although progress has been made in some special cases.

The conditions for  $V(x) = x^T P x$  to be a CQLF are equivalent to a system of linear matrix inequalities (LMIs) in  $P$ , namely  $P = P^T > 0$ ,  $(A_i^T P + P A_i) < 0$  ( $A_i^T P A_i - P < 0$ ) for  $i \in \Sigma$ .

In [26], an interactive gradient decent algorithm was proposed, which could converge to the CQLF in finite number of steps. In addition, the author showed that the convergence rate could be improved by introducing some randomness; here the convergence is in the sense of probability one. While numerical methods to solve these LMIs for a finite number of stable LTI systems have existed for some time, determining algebraic conditions (on the subsystems' state matrices) for the existence of a CQLF remains a challenging task. Since these kind of conditions should be easier to verify, and more importantly, may give us valuable insights in the stability problem of an arbitrary switching system. One long-standing goal in the field of switched systems has been to find simple algebraic conditions for existence of a CQLF for a given set of matrices. In the discrete-time case it is known by the work of Kozyakin [27], that exponential stability is not a property that can be described by finitely many algebraic constraints in the set of pairs of 2-by-2 matrices.

Now, we present some partial solutions to this general problem in continuous-time.

**Theorem 1.** [28] Let  $A_1$  and  $A_2$  be 2-by-2 Hurwitz matrices. The following conditions are equivalent:

1. there exist a CQLF for (1);
2. the matrices  $A_1 A_2$  and  $A_1 A_2^{-1}$  do not have any negative real eigenvalues.

Generalization the above algebraic condition to higher dimensional systems turns out to be difficult. In [29], necessary and sufficient algebraic conditions were derived for the

non-existence of a CQLF for an arbitrary switching systems composed of a pair of third-order systems. For a pair of  $n$ -th order systems, a necessary condition for the existence of a CQLF was derived in [30].

**Theorem 2.** Let  $A_1$  and  $A_2$  be  $n$ -by- $n$  Hurwitz matrices. A necessary condition for the existence of a CQLF is that the matrix products  $A_1 (\alpha A_1 + (1 - \alpha) A_2)$  and  $A_1 (\alpha A_1 + (1 - \alpha) A_2)^{-1}$  do not have any negative real eigenvalues for all  $0 \leq \alpha \leq 1$ .

To formulate discrete-time versions of this theorems let us introduce the following notation:

For fixed square matrices  $A_1$  and  $A_2$  define two matrix pencils

$$\begin{aligned} H(\alpha, A_1, A_2) &= \\ [0.5I - G(\alpha, A_1, A_2)]^{-1} [0.5I + G(\alpha, A_1, A_2)] \\ G(\alpha, A_1, A_2) &= \\ \alpha [0.5I + (I + A_1)^{-1}] + (1 - \alpha) [0.5I - (I + A_2)^{-1}], \end{aligned}$$

where  $\alpha \in [0, 1]$ . The matrix pencil, is said to be Schure, if the eigenvalues of the matrix for every  $\alpha \in [0, 1]$  lie inside unit circle.

**Theorem 3.** [31] Let  $A_1$  and  $A_2$  be 2-by-2 Hurwitz matrices. CQLF exists if and only if the matrix pencils  $H(\alpha, A_1, A_2)$  and  $H(\alpha, A_1, -A_2)$  are Schur matrices.

Obviously, a necessary condition for the existence of a CQLF for a switched systems is that every pair of its subsystems share a CQLF. Actually, the existence of a CQLF for every pair of subsystems may also imply the existence of a CQLF for the switched system in certain special cases, e.g., second order positive systems [32]. Unfortunately, this does not hold in general. The existence of a CQLF for a finite number of second order LTI systems was investigated in [33], and it is interesting to observe that a necessary and sufficient condition for the existence of a CQLF is that a CQLF exists for every 3-tuple of systems.

Let us return to the continuous-time case and suppose that  $\Sigma = \{A_1, \dots, A_m\}$  consists of Hurwitz matrices all in upper triangular form, then it was shown in [34], that a CQLF always exists, and furthermore that the matrix  $P$  which defines the CQLF can be chosen to be diagonal. An interesting application of this result arises when the matrices in  $\Sigma$  commute with each other. In this case there is a unitary matrix  $U$  such that  $U^T A_i U$  is in upper triangular form for each  $i = 1, \dots, m$ , and it then follows that a CQLF exists [4]. Interestin results about common quadratic Lyapunov functions are published in [35].

**3.2. Lyapunov exponents methods.** One of the basic properties of switched linear systems is that a growth rate may be defined similarly as in the case of linear time-invariant systems. The definition proceeds similarly in the continuous and discrete-time cases. There are several approaches to defining the exponential growth rate, all of which turn out to be equivalent. A trajectory based definition considers Lyapunov exponents [36–48] of individual trajectories, which are defined in the discrete-time case as

$$\lambda(x_0, \sigma) = \limsup_{m \rightarrow \infty} \|x(m, \sigma, x_0)\|^{1/m}.$$

The exponential growth rate of the switched system is then defined by the maximal Lyapunov exponent  $\sup_{\sigma} \lambda(x_0, \sigma)$ . There are numerous approaches to the computation of growth rates, either in their guise as maximal Lyapunov exponents or as joint spectral radii.

Consider the discrete system (1). Denote by  $\mathcal{D}(\Sigma)$  the set of all sequences of matrices from set  $\Sigma$ . For fixed  $d \in \mathcal{D}(\Sigma)$ ,  $d = (A(1), A(2), \dots)$  define

$$T_m^d = A(1) \dots A(m-1)A(m)$$

and

$$\bar{\rho}(d) = \limsup_{m \rightarrow \infty} \|T_m^d\|^{1/m}. \tag{6}$$

The equivalence of all norms in finite-dimensional space implies that the quantity  $\bar{\rho}(d)$  does not depend on the choice of norm. The quantity  $\ln \bar{\rho}(d)$  is known in the control theory literature as the greatest Lyapunov exponent. Properties of Lyapunov exponents were studied in [37–51], where sufficient conditions so that Lyapunov exponents characterizes exponential stability for switched systems are shown. In particular, it was shown that the system (1) is absolutely stable if and only if

$$\sup_{d \in \mathcal{D}} \bar{\rho}(d) < \alpha < 1.$$

The first result we will discuss describes the set  $\{\bar{\rho}(d) : d \in \mathcal{D}\}$  under the assumption that the matrices in  $\Sigma$  are invertible. It is not a very restrictive assumption and it is satisfied, in particular, if the discrete system is obtained by discretization of a continuous one. The set  $\{\bar{\rho}(d) : d \in \mathcal{D}\}$  is characterized in terms of the generalized spectral subradius and the generalized spectral radius of  $\Sigma$ . A closely related problem is also studied in [52]. The concept of joint spectral radius was introduced in [53] in a pure mathematical context but now it can be useful in many applications. For discussions of applications in wavelet theory, nonhomogeneous Markov processes, probabilistic automata, iterated function systems, hysteresis nonlinearities and stability of time-varying systems we refer to the papers [54–59] and the references therein.

Denote by  $\rho(A)$  the spectral radius of a matrix  $A$ . For  $m \geq 1$ ,  $\Sigma^m$  is the set of all products of matrices in  $\Sigma$  of length  $m$ ,

$$\Sigma^m = \{A_1 A_2 \dots A_m : A_i \in \Sigma, i = 1, \dots, m\}.$$

Set

$$\begin{aligned} \bar{\alpha}_m &= \sup_{A \in \Sigma^m} \|A\|, & \underline{\alpha}_m &= \inf_{A \in \Sigma^m} \|A\| \\ \bar{\beta}_m &= \sup_{A \in \Sigma^m} \rho(A), & \underline{\beta}_m &= \inf_{A \in \Sigma^m} \rho(A) \end{aligned}$$

and define:

- the joint spectral subradius

$$\hat{\rho}_*(\Sigma) = \inf_{m \geq 1} \underline{\alpha}_m^{1/m},$$

- the joint spectral radius

$$\hat{\rho}(\Sigma) = \inf_{m \geq 1} \bar{\alpha}_m^{1/m},$$

- the generalized spectral subradius

$$\bar{\rho}_*(\Sigma) = \inf_{m \geq 1} \underline{\beta}_m^{1/m},$$

- the generalized spectral radius

$$\bar{\rho}(\Sigma) = \sup_{m \geq 1} \bar{\beta}_m^{1/m}.$$

The concept of joint spectral radius was introduced in [53], and the generalized spectral radius in [55] (see also [60]). In [57] and [61] two different proofs of the equality

$$\hat{\rho}(\Sigma) = \bar{\rho}(\Sigma) \tag{7}$$

were given for the bounded set  $\Sigma$ . In [55] it was also shown that for bounded set  $\Sigma$  we have

$$\hat{\rho}(\Sigma) = \lim_{m \rightarrow \infty} \bar{\alpha}_m^{1/m} = \limsup_{m \rightarrow \infty} \bar{\beta}_m^{1/m}. \tag{8}$$

The concepts of the joint spectral subradius and the generalized spectral subradius were introduced in [32] to present conditions for Markov asymptotic stability of a discrete linear inclusion. In this paper it has been also shown that

$$\hat{\rho}_*(\Sigma) = \bar{\rho}_*(\Sigma) \tag{9}$$

for finite  $\Sigma$ . In [62] these concepts have been related to the so-called mortality problem. We say that the set of matrices  $\Sigma$  is mortal if the zero matrix can be expressed as a product of finitely many matrices from  $\Sigma$ . It appears that  $\Sigma$  is mortal if and only if  $\hat{\rho}_*(\Sigma) = 0$ . Finally, in [58] equality (9) was extended to the case of any nonempty set of matrices and it was shown that

$$\hat{\rho}_*(\Sigma) = \lim_{m \rightarrow \infty} \underline{\alpha}_m^{1/m} = \liminf_{m \rightarrow \infty} \underline{\beta}_m^{1/m} = \bar{\rho}_*(\Sigma). \tag{10}$$

From (9), (10) and the definitions of  $\rho(\Sigma)$ ,  $\rho_*(\Sigma)$  and  $\bar{\rho}(d)$  the following inequality follows

$$\rho_*(\Sigma) \leq \bar{\rho}(d) \leq \rho(\Sigma)$$

for bounded set  $\Sigma$ . Because of the equalities (7) and (9) we can introduce the following definition.

**Definition 2.** For bounded set  $\Sigma$  we will denote the common value of  $\hat{\rho}(\Sigma)$  and  $\bar{\rho}(\Sigma)$  by  $\rho(\Sigma)$  and called it generalized spectral radius. For nonempty set  $\Sigma$  we will denote the common value of  $\hat{\rho}_*(\Sigma)$  and  $\bar{\rho}_*(\Sigma)$  by  $\rho_*(\Sigma)$  and called it generalized spectral subradius.

Now we can present complete structure of the set  $\{\bar{\rho}(d) : d \in \mathcal{D}\}$ .

**Theorem 4.** Suppose that  $\Sigma$  is a bounded set of invertible matrices, then for each  $\gamma \in (\rho_*(\Sigma), \rho(\Sigma))$  there exists  $d \in \mathcal{D}(\Sigma)$  such that  $\bar{\rho}(d) = \gamma$ .

The conclusion of the Theorem is no longer true if we omit the assumption about the invertibility of matrices in  $\Sigma$ , as is shown by the simple example  $\Sigma = \{0, 1\}$ . In [63] one can find example of calculation of  $\rho(\Sigma)$  for a set of 2-by-2 matrices. Some further results in this direction contains [64]. Paper [65] deals with the case of unbounded set  $\mathcal{D}$ .

**3.3. Bohl exponent.** In case of finite set  $\Sigma$  absolute stability of system (1) can be reformulated in the following way: (1) is absolutely stable if and only if for each function  $\sigma$  and each initial condition  $x_0 \in \mathbb{R}^n$  the solution  $x(k, \sigma, x_0)$  tends to zero. It means that for each  $\sigma$  the corresponding time-varying system is asymptotically stable. For time-varying discrete linear systems of the form

$$x(k+1) = A(k)x(k), \quad (11)$$

we define transition matrix as

$$\mathcal{A}(m, k) = A(m-1)\dots A(k)$$

for  $m > k$  and  $\mathcal{A}(m, m) = I$ , where  $I$  is the identity matrix. System (11) is called uniformly exponentially stable (UES), if the transition matrix satisfies

$$\|\mathcal{A}(m, k)\| \leq cq^{m-k}, \quad (12)$$

for some constants  $c, q$ ,  $0 < q < 1$ ,  $c > 1$ . The UES of system (11) may be characterized by the discrete-time version of the Bohl exponent [66] (named generalized spectral radius in [67, 68]).

**Definition 3.** The Bohl exponent  $\beta(A)$  of system (11) is defined in the following way

$$\beta(A) = \inf \{ \beta : \exists c_\beta \geq 1, m \geq k \geq 0 \implies \|\mathcal{A}(m, k)\| \leq c_\beta \beta^{m-k} \}.$$

The role of the Bohl exponent for UES and its simplest properties are given in the next theorem taken from [68].

**Theorem 5.** System (11) is UES if and only if  $\beta(A) < 1$ . Moreover

$$\beta(A) = \limsup_{k, m-k \rightarrow \infty} \|\mathcal{A}(m, k)\|^{\frac{1}{m-k}} = \inf_{t \in \mathbb{N}} \sup_{m \in \mathbb{N}} \|\mathcal{A}(t+m, m)\|^{\frac{1}{t}}.$$

We will now investigate the problem of describing permutations  $\sigma$  of natural numbers such that, if systems (11) is UES, then the following system

$$x(k+1) = A(\sigma(k))x(k) \quad (13)$$

is also UES.

The solution of this problem is presented in [69]. Before we will present the results we introduce some notation. For two nonempty subsets  $A$  and  $B$  of  $\mathbb{N}$  we write  $A < B$  when  $a < b$  for any  $a \in A$  and  $b \in B$ . Denote by  $|A|$  cardinality of a set  $A$ , and if  $k, m \in \mathbb{N}$ ,  $k \leq m$ , then

$$I(k, m) = \{k, k+1, \dots, m\},$$

$$I'(k, m) = (k, k+1, \dots, m).$$

The sets  $I(k, m)$ ,  $I'(k, m)$  are called interval and ordered interval, respectively. We say that a set  $A \subset \mathbb{N}$  (a sequence  $(a_1, a_2, \dots, a_l)$ ) is a union of  $k$  mutually separated intervals (ordered intervals) if there exist  $k$  intervals  $I_1, \dots, I_k$  ( $k$  ordered intervals  $I'_1, \dots, I'_k$ ) such that  $A = I_1 \cup \dots \cup I_k$  ( $(a_1, a_2, \dots, a_l) = (I'_1, \dots, I'_k)$ ) and  $\text{dist}(I_i, I_j) \geq 2$

( $\text{dist}(I'_i, I'_j) \geq 2$ ) for any distinct  $i, j \leq k$ . If  $\sigma$  is a permutation of natural numbers and  $k, m \in \mathbb{N}$ ,  $k \leq m$ , then  $k_\sigma(k, m)$  ( $k'_\sigma(k, m)$ ) is such a number  $k$  that set  $\sigma(I(k, m))$  (sequence  $(\sigma(k), \dots, \sigma(m))$ ) is a union of  $k$  mutually separated intervals (ordered intervals). For linear operators  $A(k)$ ,

$A(k+1), \dots, A(m)$ ,  $k \leq m$  on  $\mathbb{R}^n$  symbol  $\prod_{i=k}^m A(i)$  denotes

$A(m)A(m-1)\dots A(k)$ . We say that a permutation  $\sigma$  preserves UES if for any UES system (11), system (13) is UES. Finally, we say that a permutation  $\sigma$  preserves Bohl exponent if for all bounded sequence  $A = (A(k))_{k \in \mathbb{N}}$

$$\beta(A) = \beta_\sigma(A),$$

where  $\beta_\sigma(A)$  is the Bohl exponent of (13). Because for all complex  $z$  we have  $\beta(zA) = |z|\beta(A)$ , then from Theorem 3.3 we see that  $\sigma$  preserves UES if and only if  $\sigma$  preserves Bohl exponent.

**Theorem 6** [69]. If

$$\lim_{k, m-k \rightarrow \infty} \frac{k'_\sigma(k, m)}{m-k} = 0, \quad (14)$$

then  $\sigma$  preserves UES.

**Theorem 7** [69]. If  $\sigma$  preserves UES, then

$$\lim_{k, m-k \rightarrow \infty} \frac{k_\sigma(k, m)}{m-k} = 0. \quad (15)$$

If the matrices  $(A(k))_{k \in \mathbb{N}}$  commute, then all the steps in the proof of Theorem 6 in [69] may be repeated with ordered interval replaced by intervals and therefore, the following statement is true.

**Corollary 1.** A permutation  $\sigma$  preserves UES of all systems (11) with commuting operators  $(A(k))_{k \in \mathbb{N}}$  if and only if it satisfies the condition (15).

The last Corollary implies, in particular, that a permutation  $\sigma$  preserves UES of all scalar systems (11) if and only if it satisfies the condition (15). Unfortunately, in a general case, neither condition (15) is sufficient nor condition (14) is necessary for permutation  $\sigma$  to preserve USE of (11).

## 4. Controllability

In the literature there are several different definitions for controllability of linear control systems (see e.g. [14]). For the continuous-time linear switched control system the most frequently used definition of controllability is recalled below.

**Definition 4** [18, 19]. System (4) is said to be controllable if for any initial state  $x_0$  and any final state  $x_f$  there exist a time  $t_f > 0$ , a switching path  $\sigma : [0, t_f] \rightarrow I$  and input  $u : [0, t_f] \rightarrow \mathbb{R}^p$  such that  $x(t_f, \sigma, x_0, u) = x_f$ .

It is obvious that if one subsystem say  $(A_k, B_k)$  is controllable, then linear switched system (4) is controllable. Hence in this paper, we shall investigate the non-trivial situation where each linear subsystem  $(A_k, B_k)$ ,  $k \in I$  is not controllable

For given matrix  $n \times p$ -dimensional matrix  $B$  denote  $\text{Im } B = \beta$  and for  $n \times n$ -dimensional matrix  $A$  and a linear subspace  $\beta \subset \mathbb{R}^n$ , let  $\Gamma_{A\beta} \subset \mathbb{R}^n$  denote the minimal

$A$ -invariant linear subspace that contains linear subspace  $B$ , i.e. [18],

$$\Gamma_A \beta = \beta + A\beta + \dots + A^{n-1}\beta.$$

This operation can be defined recursively as follows

$$\Gamma_A \Gamma_C \beta = \Gamma_A (\Gamma_C \beta).$$

Following [18] let us introduce the notations:

$$D_k = \sum_{j=0}^{j=n-1} A_k^j \text{Im } B_k \quad \text{for } k \in I,$$

where  $\text{Im } B_k \subset \mathbb{R}^n$  represents the range space of the given matrix  $B_k$ .

Let us define recursively the nested linear subspaces in the state space  $\mathbb{R}^n$  for system (4) as follows

$$V_1 = D_1 + \dots + D_m$$

$$V_{j+1} = \Gamma_{A_1} V_j + \dots + \Gamma_{A_m} V_j \quad \text{for } j = 1, 2, \dots$$

and finally

$$V = \sum_{j=1}^{j=\infty} V_j.$$

The linear space  $V$  plays very important role in controllability problems and in fact it is the controllable set for the switched linear control system (4). Hence, we have the following necessary and sufficient condition for controllability.

**Theorem 8** [18,70]. The switched linear continuous-time control system (4) is controllable if and only if

$$V = \mathbb{R}^n.$$

**Remark 1.** Computational aspects for a procedure to calculate the linear space  $V$  can be found in the paper [18]. However, it requires large computational effort if the dimensions  $n$  and  $m$  are relatively large.

Similarly as in the case of standard linear control systems [14], for linear switched control system (4) it is possible to define controllability matrix [18, 70].

$$\begin{aligned} C(A_1, \dots, A_m, B_1, \dots, B_m) = & [B_1, B_2, \dots, B_m, \\ & A_1 B_1, A_2 B_1, \dots, A_m B_1, \dots, A_1 B_m, A_2 B_m, \dots, \\ & A_m B_m, A_1^2 B_1, A_2 A_1 B_1, \dots, A_m A_1 B_1, A_1 A_2 B_1, \\ & A_2^2 B_1, \dots, A_m A_2 B_1, \dots, A_1 A_m B_m, A_2 A_m B_m, \dots, \\ & A_m^2 B_m, \dots, A_1^{n-1} B_1, A_2 A_1^{n-2} B_1, \dots, A_m A_2 A_1^{n-3} B_1, \\ & \dots, A_1 A_m^{n-2} B_m, A_2 A_m^{n-2} B_m, \dots, A_m^{n-1} B_m] \end{aligned}$$

Controllability matrix  $C(A_1, \dots, A_m, B_1, \dots, B_m)$  is a very useful tool in controllability investigation for various types linear control systems [14]. Thus we can formulate necessary and sufficient for controllability.

**Theorem 9** [18,70]. The switched linear control system (4) is controllable if and only if controllability matrix has full row rank  $n$ .

Taking into account topological properties of matrices from the above Theorem immediately follows Corollary given below.

**Corollary 2** [18]. The switched linear control system (4) is controllable if and only if

$$\text{Im } C(A_1, \dots, A_m, B_1, \dots, B_m) = \mathbb{R}^n.$$

**Remark 2.** From the above Theorem and the form of controllability matrix  $C(A_1, \dots, A_m, B_1, \dots, B_m)$  it follows that controllability concept is invariant under re-arrangement of the matrices  $A_k$  and  $B_k$  for  $k \in I$ .

**Remark 3.** For a non-switched standard linear system  $(A, B)$ , the above Theorem degenerates to the well-known Kalman controllability condition [14].

**Remark 4.** It should be notice, that similarly as in standard control systems [14] controllability concept for switched linear control systems is a dual concept for observability [18]. Algebraic observability criteria for switched linear control system (4) can be found for example in the paper [18].

**Remark 5.** For the controllable switched system (4) any initial state  $x_0$  can be transferred to each other state  $x_f$  in finite time  $t_f$ . Switching design control problem can be stated as follows: for a given any two states  $x_0$  and  $x_f$ , find a switching path  $\sigma$  and control input  $u$  to steer the system from  $x_0$  to  $x_f$  in finite time  $t_f$ . Generally, there exist many different controls making the above transfer. Moreover, it is well known (see e.g. [14–16]) that for standard linear control systems the design control problem is strongly related to so-called minimum energy control problem.

**4.1. Structural controllability.** The traditional controllability concept can be extended for so-called structural controllability, which may be more reasonable in case of uncertainties [20]. It should be pointed out, that in practice most of system parameter values are difficult to identify and are known only to certain approximations. Thus structural controllability which is independent on a specific value of unknown parameters are of particular interest. Roughly speaking, switched linear system is said to be structurally controllable if one can find a set of values for the free parameters such that the corresponding switched system is controllable in the standard sense [14, 20].

In view of the structural controllability consideration for the switched linear control system (4) the elements of all the matrices  $(A_1, B_1, A_2, B_2, \dots, A_m, B_m)$  are either fixed zero or free independent parameters. Such matrices are structured matrices. Thus, for structured matrices fixing numerically all free parameters at some particular values we obtain matrices which are called admissible numerical realization.

**Definition 5** [20]. The switched linear system (4) is said to be structurally controllable if and only if there exists at least one admissible numerical realization such that the corresponding switched linear system is controllable in the usual sense.

In order to present structural controllability condition it is necessary to introduce the algebraic concept of so-called generic rank for the structured matrix.

**Definition 6** [20]. The generic rank shortly denoted as g-rank of a structured matrix is defined to be the maximal rank that matrix achieves as a function of its free parameters.

**Theorem 10** [20]. The switched linear control system (4) is structurally controllable if and only if

$$g - \text{rank} C(A_1, \dots, A_m, B_1, \dots, B_m) = n.$$

**Remark 6.** It is well known that structural controllability conditions for the switched linear control system (4) can be also formulated using quite general theory of graphs and the concept of irreducible matrices [20].

Structural controllability of switched linear control system (4) is strongly related to numerical computations of distance from a given controllable switched linear control system to the nearest uncontrollable one [21].

First of all let us observe, that from algebraic characterization of controllability and structural controllability immediately follows that controllability is a generic property in the space of matrices defining such systems [14, 20, 21]. Therefore, the set of controllable switched systems is an open and dense subset. Hence, it is important to know how far a controllable switched system is from the nearest uncontrollable switched system. This is specially important for switched systems with matrices whose coefficients are given with some parameter uncertainty.

Explicit bound for the distance between a controllable switched linear control system (4) to the closed set of uncontrollable switched linear control system can be obtained using special norm defined for the set of matrices and singular value decomposition for structured controllability matrix [21].

**4.2. Discrete-time switched systems.** Controllability of various discrete-time linear control systems has been considered in many publications (see e.g. monograph [14] for the list of references). It is important to note, that discrete-time switched linear control system (3) is a special case of general standard linear discrete-time control systems with variable matrices. Controllability of these discrete systems was considered for example in the monograph [14] and in the papers [71] and [72]. However, for simplicity of consideration it is assumed that discrete-time switched control system (3) is reversible [17].

**Definition 7** [17]. The discrete-time switched control system (3) is said to be reversible, if all matrices  $A_i$ ,  $i = 1, 2, \dots, m$  are nonsingular.

It should be mentioned that any causal discrete-time switched system can be realized with a reversible state variable representation. Accordingly, reversible system representation is very general and applicable to a large class of systems.

**Definition 8** [17]. The discrete-time switched linear control system (3) is said to be controllable if for any initial state  $x_0$  and any final state  $x_f$  there exist a time instant

$k > 0$ , a switching path  $\sigma : [0, k - 1] \rightarrow I$  and inputs  $u : [0, k - 1] \rightarrow \mathbb{R}^p$  such that  $x(k; \sigma, x_0, u) = x_f$ .

Similarly as for continuous-time switched linear control systems (4), controllability of reversible switched linear discrete-time control system (3) can be analyzed using controllability matrix defined in previous section. Thus we have the following necessary and sufficient condition for controllability.

**Theorem 11** [17]. The reversible switched linear discrete-time control system (3) is controllable if and only if controllability matrix  $C(A_1, \dots, A_m, B_1, \dots, B_m)$  has full row rank  $n$ .

**Remark 7.** The controllability condition given in the above Theorem is pure algebraic. However, it should be pointed out that using notation and methods taken directly from linear algebra, it is possible to formulate geometric criteria for controllability of linear discrete switched system (3) similar to those in a continuous case.

**Remark 8.** It should be stressed that for a non-switched standard linear control system  $(A, B)$ , the above theorem degenerates to the well-known necessary and sufficient geometric criterion for controllability

$$\text{Im } B + A \text{Im } B + \dots + A^k \text{Im } B + \dots + A^{n-1} \text{Im } B = \mathbb{R}^n,$$

which is equivalent to the Kalman-type rank condition [14, 71, 72].

**Remark 9.** It should be notice, that similarly as in standard discrete time control systems [14, 71, 72] the controllability concept for switched linear control systems is a dual concept for observability [17]. Algebraic observability criteria for discrete switched linear control system (3) can be found for the example in the paper [17].

**Remark 10.** For the controllable discrete linear switched system (3) any initial state  $x_0$  can be transferred to each other state  $x_f$  in finite time  $t_f$ . Switching design control problem can be stated as follows: for a given any two states  $x_0$  and  $x_f$ , find a switching path  $\sigma$  and control input  $u$  to steer the system from  $x_0$  to  $x_f$  in finite time  $t_f$ . Generally, there exist many different controls making the above transfer. Moreover, it is well known (see e.g. [14, 71, 72]) that for standard linear discrete control systems the design control problem is strongly related to so-called minimum energy control problem.

## 5. Conclusions

Stability and controllability problems for different types of dynamical systems require the application of numerous mathematical concepts and methods taken directly from differential geometry, functional analysis, topology, the matrix analysis and theory of ordinary and partial differential equations and the theory of difference equations. It should be pointed out, that the state-space models of dynamical systems provide a robust and universal method for studying stability and controllability of various classes of systems. In the paper, using the state-space approach, a survey of recent results both for stability and controllability of linear

finite-dimensional stationary switched dynamical systems has been presented. Moreover, many remarks and comments on the relationships to the literature have been given and discussed.

Finally, it should be stressed, that there are numerous open problems both for stability and controllability concepts for special types of switched dynamical systems. For example, it should be pointed out, that up to present time the most literature on controllability problems has been mainly concerned with unconstrained admissible controls and without delays in the state variables or in the controls. Therefore, in the future works the special attention should be paid on these interesting open problems to obtain results similar to these from [73]. Also, it would be very interesting to extend the reported results on fractional order systems [74] or positive systems [75].

**Acknowledgements.** The research presented here was done by authors as parts of the projects funded by the National Science Centre granted according to decisions DEC-2012/05/B/ST7/00065, DEC-2012/07/B/ST7/01404 and DEC-2012/07/N/ST7/03236, respectively.

## REFERENCES

- [1] P.J. Antsaklis, J.A. Stiver, and M.D. Lemmon, "Hybrid system modeling and autonomous control systems", *Hybrid Systems*, eds. R. Grossman, A. Nerode, A. Ravn, and H. Rischel, pp. 366–392, Springer, New York, 1993.
- [2] D. Liberzon, *Switching in Systems and Control. Systems & Control: Foundations and Applications series*, Birkhauser, Boston, 2003.
- [3] Z. Sun and S.S. Ge, *Stability Theory of Switched Dynamical Systems*, Springer, London, 2011.
- [4] K.S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices", *IEEE Trans. Automat. Control* 39, 2469–2471 (1994).
- [5] J. Hespanha, S. Bohacek, K. Obratzka, and J. Lee, "Hybrid model of TCP congestion control", *Hybrid Systems: Computation and Control* 291–304 (2001).
- [6] A. Gollu and P.P. Varaiya, "Hybrid dynamical systems", *Proc. 28th IEEE Conf. Decision and Control* 1, 2708–2712 (1989).
- [7] R.W. Brockett, "Hybrid models for motion control systems", *Essays on Control Perspectives in the Theory and its Applications*, eds. H.L. Trentelman and J.C. Willems, pp. 29–53, Birkhauser, Boston, 1993.
- [8] A. Back, J. Guckenheimer, and M. Myers, "A dynamical simulation facility for hybrid systems", *Hybrid Systems*, eds. R. Grossman, A. Nerode, A. Ravn, and H. Rischel, pp. 255–267, Springer, New York, 1993.
- [9] P.P. Varaiya, "Smart cars on smart roads: problems of control", *IEEE Trans. Automat. Contr.* 38, 195–207 (1993).
- [10] T. Hagiwara and M. Araki, "Design of a stable feedback controller based on the multirate sampling of the plant output", *IEEE Trans. Automat. Contr.* 33, 812–819 (1988).
- [11] K.M. Passino, A.N. Michel, and P.J. Antsaklis, "Lyapunov stability of a class of discrete event systems", *IEEE Trans. Automat. Contr.* 39, 269–279 (1994).
- [12] M. Margaliot, "Stability analysis of switched systems using variational principles: an introduction", *Automatica* 42, 2059–2077 (2006).
- [13] R. Shorten, F. Wirth, O. Mason, K. Wulff, and Ch. King, "Stability criteria for switched and hybrid systems", *SIAM Review* 49 (4), 545–592 (2007).
- [14] J. Klamka, *Controllability of Dynamical Systems*, Kluwer, Dordrecht, 1991.
- [15] J. Klamka, "Controllability of dynamical systems", *Applied Mathematics* 9, 57–75 (2008).
- [16] J. Klamka, "Controllability of dynamical systems—a survey", *Archives of Control Sciences* 2 (XXXVIII), 281–307 (1993).
- [17] S.S. Ge, Z. Sun, and T.H. Lee, "Reachability and controllability of switched linear discrete-time system", *IEEE Trans. on Automatic Control* AC-46 (9), 1437–1441 (2001).
- [18] Z. Sun, S.S. Ge, and T.H. Lee, "Controllability and reachability criteria for switched linear system", *Automatica* 38 (5), 775–786 (2002).
- [19] Z. Sun and S.S. Ge, *Switched Linear Systems—Control and Design*, Springer, New York, 2004.
- [20] X. Liu, H. Lin, and B.M. Chen, "Structural controllability of switched linear systems", ArXiv.1106.1703v2 [cs.SY] (2012).
- [21] J. Clotet, J. Ferrer, and M.D. Magret, "Distance from a controllable switched linear system to an uncontrollable one", *2nd Meeting on Linear Algebra Matrix Analysis and Applications* 1, CD-ROM (2010).
- [22] Y. Qiao and D. Cheng, "On partitioned controllability of switched linear systems", *Automatica* 45 (1), 225–229 (2009).
- [23] T. Kaczorek, "Positive switched 2D linear systems described by the Roesser Models", *Eur. J. Control* 18 (3), 239–246 (2012).
- [24] M.S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems", *IEEE Trans. Automat. Control* 43, 475–482 (1998).
- [25] R.A. Decarlo, M.S. Branicky, S. Pettersson, B. Lennartson, and P.J. Antsaklis, "Perspectives and results on the stability and stabilizability of hybrid systems", *Proc. IEEE: Special Issue Hybrid Systems* 88, 1069–1082 (2000).
- [26] D. Liberzon and R. Tempo, "Common Lyapunov functions and gradient algorithms", *IEEE Trans. Automat. Control* 49 (6), 990–994 (2004).
- [27] V.S. Kozyakin, "Algebraic unsolvability of problem of absolute stability of desynchronized systems", *Autom. Remote Control* 51, 754–759 (1990).
- [28] R. Shorten and K. Narendra, "Necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order linear time-invariant systems", *Proc. Amer. Control Conf.* 1, 1410–1414 (1999).
- [29] C. King and R. Shorten, "A singularity test for the existence of common quadratic Lyapunov functions for pairs of stable LTI systems", *Proc. Amer. Control Conf.* 1, 3881–3884 (2004).
- [30] R. Shorten, K. Narendra, and O. Mason, "A result on common quadratic Lyapunov functions", *IEEE Trans. Automat. Control* 48 (1), 110–113, (2003).
- [31] M. Akar and K.S. Narendra, "On the existence of a common quadratic Lyapunov function for two stable second order LTI discrete-time systems", *Proc. American Control Conf.* 25–27, 2572–2577 (2001).
- [32] L. Gurvits, R. Shorten, and O. Mason, "On the stability of switched positive linear systems", *IEEE Trans. Automat. Control* 52 (6), 1099–1103 (2007).
- [33] R. Shorten and K. Narendra, "Necessary and sufficient conditions for the existence of a CQLF for a finite number of stable LTI systems", *Int. J. Adaptive Control Signal Processing* 16 (10), 709–728 (2002).



- [34] Y. Mori, T. Mori, and Y. Kuroe, "A solution to the common Lyapunov function problem for continuous time systems", *Proc. 36th Conf. on Decision and Control* 1, 3530–3531 (1997).
- [35] S. Bialas and M. Gora, *On the Existence of a Common Solution to the Lyapunov Equations*, preprints, 2013.
- [36] L. Arnold, J-P. Eckmann, and H. Crauel, *Lyapunov Exponents*, Springer, Berlin, 1991.
- [37] A. Czornik, "On the generalized spectral subradius", *Linear Algebra and its Applications* 407, 242–248 (2005).
- [38] A. Czornik and P. Jurgaś, "Set of possible values of maximal Lyapunov exponents of discrete time-varying linear system", *Automatica* 44 (2), 580–583 (2008).
- [39] A. Czornik and A. Nawrat, "On the regularity of discrete linear systems", *Linear Algebra and Its Applications* 432 (11), 2745–2753 (2010).
- [40] A. Czornik and A. Nawrat, "On new estimates for Lyapunov exponents of discrete time-varying linear systems", *Automatica* 46 (4), 775–778 (2010).
- [41] A. Czornik, "Bounds for characteristic exponents of discrete linear time varying systems", *J. Franklin Institute* 347 (2), 502–507 (2010).
- [42] A. Czornik, "On the Perron exponents of discrete linear systems", *Linear Algebra and Its Applications* 432 (1), 394–401 (2010).
- [43] A. Czornik, P. Mokry, and A. Nawrat, "On the exponential exponents of discrete linear systems", *Linear Algebra and Its Applications* 433 (4), 867–875 (2010).
- [44] A. Czornik, P. Mokry, and A. Nawrat, "On the sigma exponent of discrete linear systems", *IEEE Trans. on Automatic Control* 55 (6), 1511–1515 (2010).
- [45] A. Czornik and A. Nawrat, "Stability by the linear approximation for discrete equations", *Linear Algebra and Its Applications* 435 (4), 742–750 (2011).
- [46] A. Czornik and A. Nawrat, "On the perturbations preserving spectrum of discrete linear systems", *J. Difference Equations and Applications* 17 (1), 57–67 (2011).
- [47] A. Czornik, P. Mokry, and M. Niezabitowski, "On a continuity of characteristic exponents of linear discrete time-varying systems", *Archives of Control Science* 22, 17–27 (2012).
- [48] A. Czornik, A. Nawrat, and M. Niezabitowski, "Lyapunov exponents for discrete time-varying systems", *Advances for Intelligent Systems* 1, 29–44 (2012).
- [49] N.E. Barabanov, "Lyapunov indicator of discrete inclusions, I", *Automation and Remote Control* 49 (2), 152–157 (1988).
- [50] N.E. Barabanov, "Lyapunov indicator of discrete inclusions, II", *Automation and Remote Control* 49 (3), 283–287 (1988).
- [51] N.E. Barabanov, "Lyapunov indicator of discrete inclusions, III", *Automation and Remote Control* 49 (5), 558–565 (1988).
- [52] F. Wirth, "Dynamics of time-varying discrete-time linear systems: Spectral theory and the projected system", *SIAM J. Control and Optimization* 36 (2), 447–487 (1998).
- [53] G.C. Rota and G. Strang, "A note on the joint spectral radius", *Inag. Math.* 22, 379–381 (1960).
- [54] V.D. Blondel and J.N. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems", *Automatica* 35 (3), 479–489 (1999).
- [55] I. Daubechies and J.C. Lagarias, "Sets of matrices all infinite products of which converge", *Linear Algebra and Its Applications* 161, 227–263 (1992).
- [56] L. Gurvits, "Stability of discrete linear inclusion", *Linear Algebra and Its Applications* 231, 47–85 (1995).
- [57] M.A. Berger and Y. Wang, "Bounded semigroups of matrices", *Linear Algebra and Its Applications* 166, 21–27 (1992).
- [58] A. Czornik, "On the generalized spectral subradius", *Linear Algebra and Its Applications* 407, 242–248 (2005).
- [59] R.M. Jungers, "The joint spectral radius: theory and applications", in *Lecture Notes in Control and Information Sciences*, vol. 385, Springer, Berlin, 2009.
- [60] I. Daubechies and J.C. Lagarias, "Corrigendum/addendum to: Sets of matrices all infinite products of which converge", *Linear Algebra and Its Applications* 327, 69–83 (2001).
- [61] L. Elsner, "The generalized spectral radius theorem: An analytic-Geometric proof", *Linear Algebra and Its Applications* 220, 151–159 (1995).
- [62] V.D. Blondel and J.N. Tsitsiklis, "When is a pair of matrices mortal?", *Inform. Process. Lett.* 63, 283–286 (1997).
- [63] A. Czornik and M. Niezabitowski, "On the Lyapunov exponents of a class of the second order discrete time linear systems with bounded perturbations", *Dynamical Systems: an International J.*, DOI:10.1080/14689367.2012.748718 (2012).
- [64] A. Czornik, A. Nawrat, and M. Niezabitowski, "On the spectrum of discrete time-varying linear systems", *Nonlinear Analysis: Hybrid Systems* 9, 27–41 (2013).
- [65] A. Czornik and M. Niezabitowski, "Lyapunov exponents for systems with unbounded coefficients", *Dynamical Systems: Int. J.* 28 (2), 140–153 (2013).
- [66] J.L. Daleckii and M.G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces*, A.M.S. Providence, Rhode Island, 1974.
- [67] A. Czornik and P. Jurgaś, "Some properties of the spectral radius of a set of matrices", *Int. J. Applied Mathematics and Computer Science* 16 (2), 183–188 (2006).
- [68] K.M. Przyłuski, "Remarks on the stability of linear infinite-dimensional discrete-time systems", *J. Differ. Equations* 72 (2), 189–200 (1988).
- [69] A. Czornik and A. Nawrat, "On rearrangements of exponentially stable discrete linear systems", *Proc. 18th IFAC World Congress* 1, 13317–13319 (2011).
- [70] Z. Sun and D.Z. Zheng, "On reachability and stabilization of switched linear control systems", *IEEE Trans. on Automatic Control* AC-46 (2), 291–295 (2001).
- [71] J. Klamka, "Relative and absolute controllability of discrete systems with delays in control", *Int. J. Control* 26 (1), 65–74 (1977).
- [72] J. Klamka, "Minimum energy control of discrete systems with delays in control", *Int. J. Control*, 26(5), 737–744 (1977).
- [73] B. Sikora and J. Klamka, "On constrained stochastic controllability of dynamical systems with multiple delays in control", *Bull. Pol. Ac.: Tech.* 60 (2), 301–305 (2012).
- [74] M. Busłowicz, "Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders", *Bull. Pol. Ac.: Tech.* 60 (2), 279–284 (2012).
- [75] T. Kaczorek, "Positive fractional continuous-time linear systems with singular pencils", *Bull. Pol. Ac.: Tech.* 60 (1), 9–12 (2012).