

# Generalization of the modulating functions method into the fractional differential equations

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**Abstract.** The main aim of the paper is to generalize the modulating functions method to be useful in all models described by differential equations with fractional derivatives, if fractional differential operator is linear. The other aim is to prove that the task of parameter identification for differential equation with fractional differentials can be simplified or reduced to an integer order. The main role of modulating functions is to reduce the order of the derivative in the equation, to obtain equations without derivatives of the output signal and to eliminate the necessity of solving differential equations.

**Key words:** fractional calculus; fractional differential systems; identification methods.

## 1. Introduction

During a process of identification, identification of parameters in particular, sometimes it is difficult to estimate or find the values of parameters for the assumed model, especially when equations describing the given object are complicated.

One method of facilitating the process is to simplify the complicated equations (reducing the order of the derivative or eliminating it entirely from the equation) by using the modulating functions method [1–4]. This method was developed by Shinbrot in the early fifties [5–6], who suggested applying integral transformations to simplify the process of parameter identification in the high order non-linear dynamic systems.

The role of modulating functions is:

- to reduce the order of the derivative in the equation
- to yield equations without derivatives of the output signal
- to eliminate the necessity of solving differential equations
- to reduce the impact of measurement noises by integrating or averaging observations instead of trying to approximate their derivatives.

Identification of the systems with fractional equations is valuable when:

- a) the classical methods are not completely applicable
- b) many fields of science have examples of discoveries, when description in the classical calculus form is not sufficiently precise, whereas description in fractional calculus form is satisfying. The examples are shown in [7–15].

The paper is organized as follows.

In Sec.2 we present definitions of fractional integral and derivative, properties and assumptions for modulating function. In Sec. 3 also basic transformations are presented

In Sec. 4 we consider the concept of applying the modulating functions to reduce the order of differential equations

with fractional derivatives. Finally, in Sec. 5 we apply fractional transformations to modulating the spline function of Maletynski and Sec. 6 is devoted to conclusions.

## 2. Definitions and assumptions

**Definition 1. Riemann-Liouville fractional integral [5].** For real values  $z > 0$  and  $f(t)$ , the piecewise continuous on  $J' = (0, \infty)$  and integrable on any finite subinterval  $J = [0, \infty)$ , for  $t > 0$  the fractional integral of  $f(t)$  can be defined as

$$D^{-z}f(t) = \frac{1}{\Gamma(z)} \int_0^t (t - \varepsilon)^{z-1} f(\varepsilon) d\varepsilon, \quad (1)$$

where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

The integral interval  $(0, t)$  applies to all the examined issues. Therefore the symbolic notation of integral  $D^{-z}f(t)$  in the above formula is equivalent to  ${}_0D_t^{-z}f(t)$ .

**Definition 2. Riemann-Liouville fractional derivative [5].** Let  $\mu > 0$  and  $m$  be the smallest integer that exceeds  $\mu$  and  $z = m - \mu > 0$ . Then the fractional derivative of  $f(t)$  of the order  $\mu$  for  $t > 0$  is defined as:

$$D^\mu f(t) = D^m [D^{-z}f(t)]. \quad (2)$$

The (2) formula is derived from the  $m$ -fold differentiation of the formula (1) so as to get a fractional derivative order equal to  $\mu$ . If  $m = \mu$ , then  $z = 0$  and the formula (2) becomes a simple identity. However, if  $\mu$  is a positive integer equal to  $p$ , then the derivative  $D^p f(t)$  can exist for  $t > 0$ , even if  $f(t)$  does not fulfill the requirements of definition 1. For example: let  $f(t)$  be  $f(t) = t^{-1}$ . If  $f(t)$  has a continuous derivative of

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$p$  order on  $J$ , then this function satisfies the definition 1 and from the below formula

$$D^p f(t) = D^{p+1} \int_0^t f(\varepsilon) d\varepsilon = D^p f(t), \quad (3)$$

we can see that the formula (2) agrees with a classic definition of an ordinary derivative. The proofs of the properties given below can be found in [12] and [13].

**Property 1.** Let function  $f(t)$  fulfill the requirements of Definition 1 and let  $\mu > 0$  and  $z > 0$ . Then for every  $t > 0$  the following dependence takes place:

$$D^{-z}[D^{-\mu} f(t)] = D^{-(\mu+z)} f(t) = D^{-\mu}[D^{-z} f(t)]. \quad (4)$$

**Property 2 [13].** Let function  $f(t)$  be continuous in  $[0, T]$  interval and let  $g(t)$  be an analytical function in the  $[0, T]$  interval while  $t \in (0, T]$ . Then for  $z > 0$ , the dependence called the Leibniz formula takes place:

$$D^{-z}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{-z}{k} [D^k g(t)][D^{-z-k} f(t)]. \quad (5)$$

**Assumptions.** Let the modulating function be denoted as  $p(t)$ , further on we will shorten it to  $p$ . This function is defined explicitly and it should satisfy the three conditions below (6–9). At this stage we suggest to add a condition that the fractional derivatives in the interval  $[0, T]$  exist.

**Condition 1**

$$p(t) = 0 \quad t \notin [0, T]. \quad (6)$$

**Condition 2**

$$D^{v,z,i} p, \text{ exist } \forall (v, z, i) \text{ in the model.} \quad (7)$$

**Condition 3**

$$D^{v,z,i} p(0) = 0, \quad (8)$$

and

$$D^{v,z,i} p(T) = 0 \quad \forall (v, z, i) \text{ in the model.} \quad (9)$$

The examples of functions fulfilling conditions 1–3 (6–9) will be given in paragraph 5.

### 3. Basic transformations

The formulas presented below transform derivatives of integer order of a function  $f$  by integration by parts. They are crucial for the method of the modulating function approach. Then the process is extended to fractional derivatives. These formulas will be applied to differential equations multiplied with the chosen modulating function.

Let  $f, g, y$  be functions  $f(t), g(t), y(t)$  therefore the derivative of the product of  $f$  and  $g$  functions is expressed by a formula from definition:

$$D(fg) = fD(g) + gD(f) \quad (10)$$

the product of function  $g$  and the derivative of function  $f$  can be expressed as:

$$gD(f) = D(fg) - fD(g). \quad (11)$$

The integral of product of functions'  $f$  and  $g$  derivative is the product of these functions

$$D^{-1}(D(fg)) = fg. \quad (12)$$

Substituting (10) into (12) we obtain the classical equation for integration by parts:

$$D^{-1}(gD(f)) = fg - D^{-1}(fD(g)). \quad (13)$$

The above procedure leads to the elimination of one function derivative and incorporating it into another one if this derivative is of an integer order. Below we shall show that this trick can be extended also for fractional derivatives of function of the order  $v$ , say consider function  $f$ , which can be expressed by  $f = D^v(y)$ . The fractional derivative of function  $y$  can be expressed as follows

$$f = D(D^{-1+v}(y)). \quad (14)$$

Substituting (14) into (13) yields:

$$D^{-1}(gD(f)) = D^{-1}(gD(D^{-1+v}(y))) : \quad (15)$$

$$D^{-1}(gD(f)) = gD^{-1+v}(y) - D^{-1}(D^{-1+v}(y)D(g)). \quad (16)$$

In the case of a definite integration on  $(0, T)$ , the equality analogous to (16) can be expressed as:

$${}_0D_T^{-1}(gD(f)) = [gD^{-1+v}(y)]_0^T - {}_0D_T^{-1}(D^{-1+v}(y)D(g)) \quad (17)$$

which, after simple calculations gives:

$${}_0D_T^{-1}(gD(f)) = [g(T)D^{-1+v}(y(T))] - [g(0)D^{-1+v}(y(0))] - {}_0D_T^{-1}(D^{-1+v}(y)D(g)), \quad (18)$$

where  ${}_0D_T^{-1}$  is a definite integral on  $(0, T)$ .

If we have the derivative of a  $n$  order, where  $n$  is a positive integer i.e.  $f = D^n(y)$ , then  $n \in N$ , the integration by parts on  $[0, T]$  is as follows:

$${}_0D_T^{-1}(gD(f)) = {}_0D_T^{-1}(gD(D^{-1+n}(y))), \quad (19)$$

$${}_0D_T^{-1}(gD(f)) = [gD^{-1+n}(y)]_0^T - {}_0D_T^{-1}(D^{-1+n}(y)D(g)) \quad (20)$$

which results in:

$${}_0D_T^{-1}(gD(f)) = [g(T)D^{-1+n}(y(T))] - [g(0)D^{-1+n}(y(0))] + {}_0D_T^{-1}(D^{-1+n}(y)D(g)) \quad (21)$$

Now we can present the fractional integration of the two functions  $f(t)$  and  $g(t)$ . Basing on Theorem 2 of the Leibniz formula (5) substituting fractional derivative of function  $y(t)$  for  $f(t)$  yields:

$$D^{-v}(gf) = D^{-v}(gD^v(y)) \quad (22)$$

in the result we have:

$$D^{-v}(gf) = \sum_{k=0}^{\infty} \binom{-v}{k} [D^k(g)] \cdot [D^{-v-k}(D^v(y))]. \quad (23)$$

## Generalization of the modulating functions method into the fractional differential equations

Let  $E$  be an operator representing a shortened denotation of:

$$E_v = \sum_{k=0}^{\infty} \binom{-v}{k}. \quad (24)$$

Simplifying the equation (23) and applying the  $E$  operator (24) the following dependence is yielded:

$$D^{-v}(gf) = E_v [D^k(g)] \cdot [D^{-k}(y)]. \quad (25)$$

As can be seen from the equation (25), the fractional derivative was entirely eliminated and replaced by a sum of the product of two functions from which one is the derivative of  $g(t)$  of the order  $k$  and the other one is the integral of  $y(t)$  of the order  $k$ .

Similarly, the fractional integration of the product of a two functions when one of them is the derivative of the order  $n$  where  $n \in N$ , can be carried out. Therefore:

$$D^{-v}(gf) = D^{-v}(gD^n(y)) = D^{-v}(D^n(y)g), \quad (26)$$

$$\begin{aligned} D^{-v}(gf) &= D^{-v}(D^n(y)g) = \\ &= E_v [D^k(D^n(y))] \cdot [D^{-v-k}(g)] \end{aligned} \quad (27)$$

and after simplifying we obtain:

$$D^{-v}(gf) = E_v [D^{k+n}(y)] \cdot [D^{-v-k}(g)]. \quad (28)$$

**Lemma 1.** If functions  $y(t)$  and  $g(t)$  are continuous on  $(0, T)$  interval and  $g(t)$  has a derivative of the order  $k + 1$ , the following dependence holds:

$$\begin{aligned} {}_0D_T^{-1}(D^{-v}(gD^v(y))) &= E_v \left( [(D^{-k-1}(y)) \cdot (D^k(g))]_0^T \right. \\ &\quad \left. - {}_0D_T^{-1} [(D^{-k-1}(y)) \cdot (D^{k+1}(g))] \right). \end{aligned} \quad (29)$$

The above dependence is obtained by integrating the formula (25) in the  $(0, T)$  interval according to the (13). It results in:

$$\begin{aligned} {}_0D_T^{-1}(D^{-v}(gf)) &= {}_0D_T^{-1}(D^{-v}(gD^v(y))) = \\ &= E_v {}_0D_T^{-1} [(D^k(g)) \cdot (D^{-k}(y))]. \end{aligned} \quad (30)$$

Converting the following expression:

$$\begin{aligned} {}_0D_T^{-1}(D^{-v}(gD^v(y))) &= \\ &= E_v {}_0D_T^{-1} [(D^{-k}(y)) \cdot (D^k(g))] = \\ &= E_v {}_0D_T^{-1} [D(D^{-k-1}(y)) \cdot (D^k(g))]. \end{aligned} \quad (31)$$

We finally obtain the (29) dependence.

**Lemma 2.** If functions  $y(t)$  and  $g(t)$  are continuous in  $(0, T)$  interval and  $g(t)$  satisfies Definitions 1 (1) and 2 (2), and  $y(t)$  has a derivative of the  $k + n - 1$  order, then the following is true:

$$\begin{aligned} {}_0D_T^{-1}(D^{-v}(D^n(y)g)) &= \\ &= E_v \left( [(D^{k+n-1}(y)) \cdot (D^{-v-k}(g))]_0^T - \right. \\ &\quad \left. - {}_0D_T^{-1} [(D^{k+n-1}(y)) \cdot (D^{-v-k+1}(g))] \right). \end{aligned} \quad (32)$$

This is obtained by integrating formula (18) in the  $(0, T)$  interval according to (13). It results in:

$$\begin{aligned} {}_0D_T^{-1}(D^{-v}(fg)) &= {}_0D_T^{-1}(D^{-v}(D^n(y)g)) = \\ &= E_v {}_0D_T^{-1} [(D^{k+n}(y)) \cdot (D^{-v-k}(g))]. \end{aligned} \quad (33)$$

Expanding integration by parts yields:

$$\begin{aligned} E_v {}_0D_T^{-1} [(D^{k+n}(y)) \cdot (D^{-v-k}(g))] &= \\ &= E_v {}_0D_T^{-1} [D(D^{k+n-1}(y)) \cdot (D^{-v-k}(g))] \end{aligned} \quad (34)$$

which gives the expression (32) in the effect.

#### 4. Concept of applying the modulating functions to reduce the order of differential equations with fractional derivatives

Assumptions were made mainly by Shinbrot as stated in [5, 6], among others, herein will be extended to the fractional derivatives case. Therefore, using a simple model with linear parameters we are going to illustrate the mechanism of fractional transformations.

A linear model with respect to parameters can be expressed as:

$$a_v D^v y + \sum_{i=0}^n a_i D^i y = b_z D^z u + \sum_{i=0}^m b_i D^i u, \quad (35)$$

where  $m \leq n$  and  $v, z \in (0, 1)$ ; for  $m = n = 0$  the inequality  $z \leq v$  shall be true.

Applying modulating function  $p(t)$  to the model (35) along with a suitable transformation of model equation should allow, besides reduction of the order of integer derivatives,

- 1° to reduce the order of fractional derivatives while coming to fractional integral,
- 2° to eliminate fractional derivatives and integrals from equation.

Therefore, the two following cases can be considered:

**1° Reduction of fractional derivatives.** It can happen, under some circumstances, that a fractional derivative of an input function  $u(t)$  or an output function  $y(t)$  does not exist, but it is reasonably easy to calculate their fractional integrals. Therefore by applying a modulating function to the model and then reducing the order of derivatives by integration of integer order it is possible to eliminate the fractional derivative of functions  $u(t)$  and  $y(t)$  from the equation, substituting it with fractional integral of these functions.

Referring to the linear model with regard to parameters equation (35), both sides of the equation can be multiplied by a selected modulating function  $p(t)$ .

$$a_v p D^v y + \sum_{i=0}^n a_i p D^i y = b_z p D^z u + \sum_{i=0}^m b_i p D^i u. \quad (36)$$

Then, we integrate by parts both sides of the equation by integral of an integer order in  $[0, T]$  interval. This allows to formulate the following theorem:

**Theorem 1.** If  $y(t)$  is the solution of differential equation of fractional derivatives as follows

$$a_v D^v y + \sum_{i=0}^n a_i D^i y = b_z D^z u + \sum_{i=0}^m b_i D^i u \quad \{35\}$$

and if  $p(t)$  fulfills conditions 1–3 (6–9), the following equation is also true:

$$\begin{aligned} & a_v \left[ [pD^{-1+v}(y)]_0^T - {}_0D_T^{-1}(D^{-1+v}(y)D(p)) \right] + \\ & + \sum_{i=0}^n a_i \left[ [pD^{-1+i}(y)]_0^T - {}_0D_T^{-1}(D^{-1+i}(y)D(p)) \right] = \\ & = b_z \left[ [pD^{-1+z}(y)]_0^T - {}_0D_T^{-1}(D^{-1+z}(y)D(p)) \right] + \\ & + \sum_{i=0}^m b_i \left[ [pD^{-1+i}(u)]_0^T - {}_0D_T^{-1}(D^{-1+i}(u)D(p)) \right]. \end{aligned} \quad (37)$$

The proof of the theorem is a result of transformations (17) applied to fractional transformations and transformations (20) used to integer ones carried out earlier. Assuming that  $p(t) = g(t)$  and conditions 1–3 (6–9) are fulfilled, the fractional derivatives of functions  $y(t)$  and  $u(t)$  are eliminated. What is more, we assumed that functions subjected to fractional differentiation (derivative, integral of these functions) are in accordance with the Definition 1 of fractional integral (1) and the Definition 2 of fractional derivative (2).

Assuming that  $a_0 = 1$  in the model, the number of parameters to be identified is  $n + m + 3$ , including parameters of ‘fractional’ elements. As a result, at least the same number of linearly independent equations has to be generated in order to calculate all the parameters [3]. Shifting the modulating function by discrete time intervals like  $\Delta T$ , additional equations may be generated. The first modulation acts in  $[0, T]$  interval, the second in  $[\Delta T, T + \Delta T]$ , the third in  $[2\Delta T, T + 2\Delta T]$  interval and so on. We notice that equations based on (35) allowing to identify parameters should be linearly independent. When the number of generated equations is higher than the number of parameters to be identified in a model, a least squares method can be applied.

## 2° Elimination of fractional derivatives and integrals.

In the second case we can go further eliminating completely all the ‘fractional’ elements from functions  $u(t)$  and  $y(t)$ , with the help of modulating function which will take over the workload from fractional elements.

**Theorem 2.** If  $y(t)$  is a solution of the following differential equation with fractional derivatives:

$$a_v D^v y + \sum_{i=0}^n a_i D^i y = b_z D^z u + \sum_{i=0}^m b_i D^i u \quad \{35\}$$

and  $p(t)$  fulfills conditions 1–3 (6–9), the following equation is also true:

$$\begin{aligned} E_z E_v [a_v D^{-z-r+k} p D^{r-k} y + E_n D^{-z-r-v-k} p D^{r+i+k} y] = \\ = E_z E_v [b_z D^{-v-k} p D^{-r+k} u + E_m D^{-z-r-v-k} p D^{r+i+k} u], \end{aligned} \quad (38)$$

where, let for simplicity reasons:

$$\begin{aligned} E_v &= \sum_{k=0}^{\infty} \binom{-v}{k}; & E_z &= \sum_{r=0}^{\infty} \binom{-z}{r}; \\ E_n &= \sum_{i=0}^n a_i; & E_m &= \sum_{i=0}^m b_i. \end{aligned}$$

Proof of the theorem.

Conducting the fractional integration of the order e.g.  $v$  of both sides of equation (36), according to expression (25), the left side of equation will be free of ‘fractions’ in the result. Therefore:

$$\begin{aligned} a_v D^{-v} (p D^v y) + E_n D^{-v} (p D^i y) = \\ = b_z D^{-v} (p D^z u) + E_m D^{-v} (p D^i u) \end{aligned} \quad (39)$$

what can be expressed as:

$$\begin{aligned} E_v [a_v D^k p D^{-k} y + E_n D^{-v-k} p D^{i+k} y] = \\ = E_v [b_z D^{-v-k} p D^{z+k} u + E_m D^{-v-k} p D^{i+k} u]. \end{aligned} \quad (40)$$

Then by integrating over  $z$  and applying the above procedure we obtain the equation (38) without fractional derivatives and integrals of functions  $u(t)$  and  $y(t)$ .

$$\begin{aligned} D^{-z} E_v [a_v D^k p D^{-k} y + E_n D^{-v-k} p D^{i+k} y] = \\ = D^{-z} E_v [b_z D^{-v-k} p D^{z+k} u + E_m D^{-v-k} p D^{i+k} u]. \end{aligned} \quad (41)$$

As the result of the above transformations the derivatives of functions  $u(t)$  and  $y(t)$  became of the higher  $k + r$  order. Because the stimulating signals and their responses derivatives of such high order might not exist, we should apply the method of integration of an integer order so as to eliminate the derivatives of these orders. Then, similarly like in the first case, the  $n+m+3$  of linearly independent equations have to be obtained to find the desired parameters.

The second method of transformation is very complicated. The modulating function has to meet numerous requirements which might cause the difficulties to find one. To avoid large scale mathematical complications it is better, in some cases, to use the 1st case of transformations applying stimulating signals that do not cause problems with fractional integration. The same concerns the output functions.

## 5. Fractional transformations applied to modulating spline function of Maletynski

In this paragraph the transformations of fractional differential equations with the use of modulating spline function of Maletynski are discussed. The Maletynski modulating spline function advantage over the other modulating functions is its generality in applications [1, 2].

Generalization of the modulating functions method into the fractional differential equations

As it was stated above, the main role of modulating functions is to enable the integral transformations usage on signals and their derivatives in order to, among others, minimize the noise in signal. In many cases [3] the modulating functions were identified experimentally so as to comply three conditions (6–9) (disregarding fractional cases). Maletynski was the first one who documented an analytical introduction of a set of modulating function to a single input – single output systems [3].

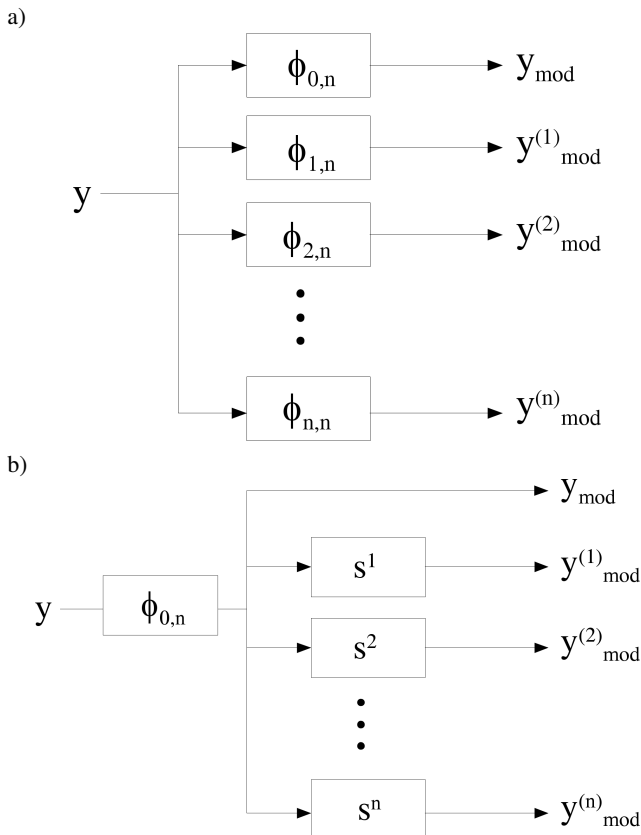


Fig. 1. The two approaches to spline modulating application in a signal

Modulating functions can be applied to a signal in two ways [3] (Fig. 1a, b):

- both groups of modulating functions are applied simultaneously, each of functions being a derivative of the previous one, beginning from the derivative of order zero or
- modulating function of order zero is applied to signal which is then differentiated progressively. Every differentiation gives the modulated signal in the result.

The first approach is useful for illustrating the idea and definition of modulating functions. The second way is helpful in analysis of modulating functions applied for filtration. For further applications the functions are grouped together. Each group consists of modulating functions and their derivatives. As a rule, each group consists of  $n + 1$  elements, where  $n$  is an order of the modulating function:

– The group of modulating functions:

$$\phi_n = \{\phi_{n,j}\}, \quad \text{where } j = 0, 1, \dots, n. \quad (42)$$

According to the rule the modulating function of the order  $n$  is the first element in the group.

– Modulating function:

$$\phi_{n,0} = \phi_n. \quad (43)$$

According to definition (42), the first index indicates the order and the second the number of differentiations of modulating function.

The axiomatic definition of the group of spline modulating functions based on the largest  $n$ -th order derivative of modulating function of the  $n$  order is given

$$\phi_{n,n}(t) = \sum_{i=0}^n (-1)^i \binom{n}{i} \delta(iT - t). \quad (44)$$

The group is supplemented by the successive ( $k = n - j$ ) times integrations

$$\phi_{n,j}(t) = \int_0^{nT} \dots \int_0^{nT} \phi_{n,n}(t) dt^k \quad \text{where } k = 0, 1, \dots, n, \quad (45)$$

where  $n$  – the first parameter defines the order of the modulating function,  $T$  – the second parameter defines the characteristic time,  $\delta$  – the Dirac delta function.

The spline modulating functions have only two parameters, the order and the characteristic time  $T$ . The order of function is related to the order of the model and usually we choose one order higher than the order of a model.

Using the equation (45) we can easily match a modulating function with a model order by further integrations of basic function which is a modified Dirac delta function. Therefore, what is left is to find how such group of functions behaves during differentiation or integration of fractional order.

The definition of a step function implies that it can be described as follows:

Step function:

$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad (46)$$

If shifted in time, it yields

$$1(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases} \quad (47)$$

The Dirac delta function  $\delta(t)$  is:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad (48)$$

and similarly shifted in time by  $t_0$ :

$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0 \end{cases} \quad (49)$$

after integration on  $(t_1, t_2)$  interval it yields following:

$$\int_{t_1}^{t_2} \delta(t - t_0) = \begin{cases} 1 & t_0 \in (t_1, t_2) \\ 0 & t_0 \notin (t_1, t_2) \end{cases} \quad (50)$$

In general, we obtain a step function:

$$\int_{-\infty}^t \delta(\tau) d\tau = 1(t). \quad (51)$$

Which after shifting in time yields:

$$\int_{-\infty}^t \delta(\tau - t_0) d\tau = \int_{-\infty}^{t-t_0} \delta(\tau) d\tau = 1(t - t_0). \quad (52)$$

If  $F(t)$  is a primitive function of type  $1(t)$  then

$$\frac{d}{dt} 1(t) = \delta(t), \quad (53)$$

$$\frac{d}{dt} 1(t - t_0) = \delta(t - t_0). \quad (54)$$

Based on the above transformations, a similar procedure can be applied to fractional integration.

Taking advantage of the Property 2 (5) we may write:

$$\begin{aligned} D^{-v}(1) &= D^{-v}(D^{-1}(\delta)) = \\ &= D^{-1}(D^{-v}(\delta)) = A, \end{aligned} \quad (55)$$

where, based on transformation given in [13]:

$$A = \frac{1}{\Gamma(v+1)} t^v. \quad (56)$$

Then differentiating both sides of the equation (56):

$$D[D^{-1}(D^{-v}(\delta))] = D[A] \quad (57)$$

we obtain a fractional integral of function  $\delta$  of the order  $v$ :

$$D^{-v}(\delta) = \frac{1}{\Gamma(v)} t^{v-1} \quad (58)$$

and  $k$  times integration of fractional integral of the order  $v$  yields [13]

$$D^{-v-k}(1) = \frac{1}{\Gamma(v+k+1)} t^{v+k} \quad (59)$$

for  $k \geq 0$ .

Derivative of order  $v$  of step function is as follows:

$$D^v(1) = \frac{1}{\Gamma(1-v)} t^{-v}. \quad (60)$$

The extended transformations of the above functions can be found in [7, 13, 14].

The interval where these transformations are determined is coincide with the interval of validity of the 'classical transformations' of modulating functions determined by Maletynski. However, it should be noticed that all transformations require the existence of integrals and derivatives of fractional order

described by definitions 1 and 2 (1-2). Therefore the equation (44) transformed by fractional integration looks like following

$$D^{-v}(\phi_{n,n}) = E \frac{1}{\Gamma(v)} (iT - t)^{v-1}, \quad (61)$$

where the operator  $E$  is expressed by:

$$E = \sum_{i=0}^n (-1)^i \binom{n}{i} \quad (62)$$

## 6. Conclusions

Through the history many scientists, like e.g. Shinbrot, Preisig and Rippin [4, 5] have been developing the modulating function methods. In this paper we presented the generalization of these methods for fractional order systems. The aim of it was to simplify orders of fractional equations so as to facilitate calculations, for example to find parameters of identifications of PVDF and Bismuth Oxide electrical models [8, 9, 11].

Basing on above and formulated in the paper theorems, beneath we propose the generalized algorithm for fractional systems applications based on modulating functions:

1. Determination of the model order for the examined system (considering fractional elements, if they exist).
2. Selection of the modulating function or a set of such functions that could take it upon oneself the fractional differentiating or integrating.
3. Depending on the model complexity, application of one of the following cases:
  - a) reduction of fractional derivatives,
  - b) elimination of fractional derivatives and integrals,
4. Application of the methods to calculate fractional derivatives or integrals from modulating functions.
5. Substitute measured  $u(t)$  and  $y(t)$  and then calculation of the model equation parameters using traditional methods.

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