10.24425/acs.2018.125482

Archives of Control Sciences
Volume 28(LXIV), 2018
No. 4, *pages 527–549*

Development of dual hesitant fuzzy prioritized operators based on Einstein operations with their application to multi-criteria group decision making

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The purpose of this article is to develop a multicriteria group decision making (MCGDM) method in dual hesitant fuzzy (DHF) environment by evaluating the weights of the decision makers from the decision matrices using two newly defined prioritized aggregation operators based on score function to remove the inconsistencies in choosing the best alternative. Prioritized weighted averaging operator and prioritized weighted geometric operator based on Einstein operations are described first for aggregating DHF information. Some of their desirable properties are also investigated in details. A method for finding the rank of alternatives in MCGDM problems with DHF information based on priority levels of decision makers is developed. An illustrative example concerning MCGDM problem is considered to establish the application potentiality of the proposed approach. The method is efficient enough to solve different real life MCGDM problems having DHF information.

Key words: multi-criteria group decision-making, aggregation operator, dual hesitant fuzzy numbers, Einstein operations, prioritized weighted averaging operator, prioritized weighted geometric operator

1. Introduction

Theory of Fuzzy sets (FSs) [31] are widely and successfully applied in all areas of real life decision making problems to handle vagueness or possibilistic imprecisions. After introduction of FSs, several extensions are developed, such as type-2 FSs (T2FSs) [1–3, 8, 10], fuzzy multisets [10], interval-valued FSs [32], etc. As a generalization of FSs, Atanassov presented the concept of intuitionistic FS (IFS) [17] using two characteristic functions representing the degree of membership and the degree of non-membership of elements of the universal

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Rceived 02.05.2018.

set to the IFS. As like FSs, in IFSs several variants are also found in the form of intuitionistic linguistic FSs [7], interval-valued IFSs [3, 4, 12, 22], etc.

In real-life applications, when decision makers are confused with assigning exact preference information using FSs or IFSs, the concept of hesitant FSs (HFSs) came into the literature [15, 16] as a new generalization of FSs as well as IFSs. HFSs deal with the difficulties of establishing a common membership degree not because of a margin of error (IFSs), or some possibility distribution values (T2FS), but have a set of possible values. Torra [15] provided a definition corresponding to the envelope of HFSs. HFS has received a considerable attention to the researchers and is applied to various fields of decision-making [14, 23]. Xia and Xu [21] developed several series of aggregation operators for hesitant fuzzy information and discussed the relationships among them. In the context of multi-criteria decision-making (MCDM), Wei et al., [20] proposed hesitant fuzzy linguistic arithmetic aggregation operators. Based on the idea of prioritized aggregation operators [26] Wei [19] defined some prioritized aggregation operators for aggregating hesitant fuzzy information and then applied them to develop models for hesitant fuzzy multiple attribute decision making (MADM) problems in which the attributes are in different priority level.

Zhu et al. [34] proposed dual hesitant fuzzy (DHF) set (DHFS) by considering several possible values for the membership as well as non-membership degrees. Thus, DHFSs can take much more information than HFSs given by decision makers into account in MADM. Wei and Lu [18] developed Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in MADM. Inspired by generalized ordered weighted average operator [25], Yu and Li [29] proposed some generalized aggregation operators for DHFEs. Different MADM theories and methods under DHF environments are developed using those aggregation operators. All the developed methods are under the assumption that the attributes are at the same priority level. However, in real and practical MADM situation, the attributes may have different priority levels. To overcome this drawback, in this paper, DHF prioritized weighted average (DHFPWA) operator and DHF prioritized weighted geometric (DHFPWG) operator are proposed and some of their properties have been discussed.

2. Some basic concepts and operations

In this section, some basic concepts, which are essential to develop the proposed methodology, are described.

Definition 1 [1] *Let a set X be fixed. An IFS α on X is represented in terms of two functions $\mu_\alpha\colon X \to [0,1]$ and $\nu_\alpha\colon X \to [0,1]$, and having the form $\alpha = \{\langle x, \mu_\alpha, \nu_\alpha(x)\rangle | x \in X\}$ with the condition $0 \leqslant \mu_\alpha(x) + \nu_\alpha(x) \leqslant 1$, for all $x \in X$,* *where $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of x in α.*

For convenience, Xu and Yager [24] called $\alpha = (\mu_\alpha, \nu_\alpha)$ an intuitionistic fuzzy number (IFN), where $\mu_\alpha \in [0,1]$, $\nu_\alpha \in [0,1]$ and $0 \leqslant \mu_\alpha + \nu_\alpha \leqslant 1$.

For any IFN $\alpha = (\mu_\alpha, \nu_\alpha)$, a score of α can be evaluated by a score function s [6] as

$$s(\alpha) = \mu_\alpha - \nu_\alpha \qquad \text{where} \quad s(\alpha) \in [-1,1]. \tag{1}$$

Definition 2 [15–16] *Let X be a universal set. A HFS A defined on X is represented by a function h_A that returns a subset of $[0,1]$ when it is applied to X.*

For convenience, Xia and Xu [21] represented the HFS A by using the mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle \,|x \in X\},\tag{2}$$

where $h_A(x)$ is a set of several different values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set A. Xia and Xu [21] called $h = h_A(x)$ a hesitant fuzzy element (HFE).

Definition 3 [34] *Let X be a fixed set, a DHFS D defined on X is represented as:*

$$D = \{\langle x, h(x), g(x)\rangle \,|x \in X\},\tag{3}$$

where $h(x)$ and $g(x)$ are hesitant fuzzy elements, denoting respectively the membership and non-membership degree of the element x to D, with the conditions:

$$0 \leqslant \gamma, \;\; \tau \leqslant 1 \quad \textit{with} \quad 0 \leqslant \gamma^+ + \tau^+ \leqslant 1,$$

where $\gamma \in h(x) \subseteq [0,1]$, $\tau \in g(x) \subseteq [0,1]$ and $\gamma^+ = \max\{h(x)\}$, $\tau^+ = \max\{g(x)\}$.

For convenience $\langle h(x), g(x)\rangle$ is called the DHF element (DHFE) and is denoted as

$$\widetilde{\alpha} = (h, g).$$

2.1. Einstein operations

The set theoretical operators play an important role to aggregate different fuzzy information. Since the inception of fuzzy set theory, starting from Zadeh's operator, min and max, many other operators introduced in the literature. All types of the operators were included in the general concepts of the t-norms and t-conorms, which satisfy the requirement of the conjunction and disjunction operators, respectively.

There are various t-norm and t-conorm families available in the literature. Einstein operators include the Einstein product $\otimes_\varepsilon$ and Einstein sum $\oplus_\varepsilon$, which are examples of t-norm and t-conorm, respectively. The Einstein operators are defined as follows [11]:

$$a \oplus_{\varepsilon} b = \frac{a+b}{1+a\cdot b}, \quad a \otimes_{\varepsilon} b = \frac{a\cdot b}{1+(1-a)(1-b)} \quad \text{for all } (a,b)\in[0,1]^2. \quad (4)$$

Based on the concepts of Einstein operators Zhao et al. [33] introduced different operations on DHFEs as follows:

Definition 4 [33] *Let $\widetilde{\alpha}_1 = (h_1, g_1)$, $\widetilde{\alpha}_2 = (h_2, g_2)$ and $\widetilde{\alpha} = (h, g)$ be three DHFEs. Then*

$$\text{(i)} \quad \widetilde{\alpha}_1 \oplus_{\varepsilon} \widetilde{\alpha}_2 = \left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2} \frac{\gamma_1+\gamma_2}{1+\gamma_1\gamma_2}, \bigcup_{\tau_1\in g_1,\tau_2\in g_2} \frac{\tau_1\tau_2}{1+(1-\tau_1)(1-\tau_2)} \right);$$

$$\text{(ii)} \quad \widetilde{\alpha}_1 \otimes_{\varepsilon} \widetilde{\alpha}_2 = \left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2} \frac{\gamma_1\gamma_2}{1+(1-\gamma_1)(1-\gamma_2)}, \bigcup_{\tau_1\in g_1,\tau_2\in g_2} \frac{\tau_1+\tau_2}{1+\tau_1\tau_2} \right);$$

$$\text{(iii)} \quad \lambda\widetilde{\alpha} = \left(\bigcup_{\gamma\in h} \frac{(1+\gamma)^\lambda-(1-\gamma)^\lambda}{(1+\gamma)^\lambda+(1-\gamma)^\lambda}, \bigcup_{\tau\in g} \frac{2\tau^\lambda}{(2-\tau)^\lambda+\tau^\lambda} \right), \quad \lambda>0;$$

$$\text{(iv)} \quad \widetilde{\alpha}^\lambda = \left(\bigcup_{\gamma\in h} \frac{2\gamma^\lambda}{(2-\gamma)^\lambda+\gamma^\lambda}, \bigcup_{\tau\in g} \frac{(1+\tau)^\lambda-(1-\tau)^\lambda}{(1+\tau)^\lambda+(1-\tau)^\lambda} \right), \quad \lambda>0.$$

2.2. Prioritized operators

The prioritized operators play also an important role in solving many MCDM problems. The prioritized averaging (PA) operator, introduced by Yager [26], is defined in the following manner:

Definition 5 [26] *Let $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criteria and that there is a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ \ldots \succ C_n$, indicate criteria C_j has a higher priority than C_k if $j<k$. The value $C_j(x)$ is the performance of any alternative x under criteria C_j, and satisfies $C_j(x)\in[0,1]$. If*

$$PA(C_j(x)) = \sum_{j=1}^{n} w_j C_j(x), \quad (5)$$

where $w_j = \dfrac{T_j}{\sum_{j=1}^{n} T_j}$, $T_j = \prod_{k=1}^{j-1} C_k(x)$ $(j=2,\ldots,n)$, $T_1 = 1$. Then PA is called the PA *operator.*

In the following Sections, the methodological development of the paper is incorporated.

At first a new score function of DHFEs is introduced. In this context it is to be pointed out that a score function defined by Zhu et al. [34] is already exist in the literature. But, the drawback of that approach is that the score value becomes negative when average of membership degree is less than the average of non-membership degree.

Based on the concepts of score functions of DHFEs, Einstein operators and prioritized operators, dual hesitant fuzzy aggregation operators are defined. The defined operators are then used to solve a MCDM problem.

The methodological developments are described subsequently.

3. Score function of a dual hesitant fuzzy element (DHFE)

A new score function is defined in this section to find the ordering of DHFEs.

Definition 6 *Score function of DHFE is defined as*

$$s(\alpha) = \frac{1 + \sum_{\gamma \in h} \frac{\gamma}{l(h)} - \sum_{\tau \in g} \frac{\tau}{l(g)}}{2} \tag{6}$$

and the accuracy function of DHFE is described as follows

$$a(\alpha) = \sum_{\gamma \in h} \frac{\gamma}{l(h)} + \sum_{\tau \in g} \frac{\tau}{l(g)} . \tag{7}$$

where $l(h)$ and $l(g)$ represents the number of elements in h and g, respectively.

For comparison of DHFEs the following conditions are to be satisfied.

Let α_1 and α_2 be two DHFEs

1. If $s(\alpha_1) > s(\alpha_2)$ then $\alpha_1 > \alpha_2$;
2. If $s(\alpha_1) = s(\alpha_2)$ then
   if $a(\alpha_1) > a(\alpha_2)$ then $\alpha_1 > \alpha_2$; if $a(\alpha_1) = a(\alpha_2)$ then $\alpha_1 = \alpha_2$.

4. Development of Dual Hesitant fuzzy aggregation operator based on prioritized operators

Based on the score function, Einstein operations and prioritized operators as defined above, a dual hesitant fuzzy prioritized Einstein aggregation operator is defined as follows.

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Definition 7 *Let $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collections of* DHFE*s and $w = (w_1, w_2, \ldots, w_n)$ be the weight vectors of $\widetilde{\alpha}_i$, where*

$$w_i = \frac{T_i}{\sum_{i=1}^{n} T_i} \quad \textit{and} \quad T_i = \prod_{k=1}^{i-1} s(\widetilde{\alpha}_k) \quad (i = 2, \ldots, n), \;\; T_1 = 1, \tag{8}$$

and $s(\widetilde{\alpha}_i)$ is the score of DHFE $\widetilde{\alpha}_i$. If DHFPEWA$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \oplus_{\varepsilon}{}_{i=1}^{n} w_i \alpha_i$ *then* DHFPEWA *is called a dual hesitant fuzzy prioritized Einstein weighted averaging operator.*

Theorem 1 *Let $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collections of* DHFE*s, then the aggregated value by using DHFPEWA operator is also a DHFE and*

$$\mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \oplus_{\varepsilon}{}_{i=1}^{n} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_i \right)$$
$$= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \frac{\prod_{i=1}^{n} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.$$
$$\left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_n \in g_n} \frac{2 \prod_{i=1}^{n} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right). \tag{9}$$

Proof. Using the mathematical induction method, the theorem will be proved.

The theorem is obvious for $n = 1$.

We assume that theorem is true for $n = p$, we shall prove that it is true for $n = p + 1$.

For $n = p$, we have

$$\mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p) = \oplus_{\varepsilon}{}_{i=1}^{p} \left(\frac{T_i}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_i \right)$$
$$= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots\ldots, \gamma_p \in h_p} \frac{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.$$
$$\left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_p \in g_p} \frac{2 \prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{p} (2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{p} T_i}} + \prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right).$$

Now when $n = p+1$,

$$\mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p, \widetilde{\alpha}_{p+1})$$

$$= \mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p) \oplus_{\varepsilon} \left(\frac{T_{p+1}}{\sum_{i=1}^{n} T_i} \widetilde{\alpha}_{p+1} \right)$$

$$= \mathrm{DHFPEWA}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_p) \oplus_{\varepsilon} \left(\bigcup_{\gamma_{p+1} \in h_{p+1}} \frac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}, \bigcup_{\tau_{p+1} \in g_{p+1}} \frac{2\tau_{p+1}^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} + \tau_{p+1}^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}} \right)$$

$$= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_p \in h_p} \frac{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_p \in g_p} \frac{2\prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{p} (2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{p} (\tau_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right) \oplus_{\varepsilon}$$

$$\left(\bigcup_{\gamma_{p+1} \in h_{p+1}} \frac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}, \bigcup_{\tau_{p+1} \in g_{p+1}} \frac{2(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} + (\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}} \right)$$

$$= \left(\bigcup_{\substack{\gamma_1 \in h_1, \\ \gamma_2 \in h_2, \ldots, \\ \gamma_p \in h_p, \\ \gamma_{p+1} \in h_{p+1}}} \frac{\dfrac{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} + \dfrac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}}{1 + \dfrac{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod\limits_{i=1}^{p} (1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod\limits_{i=1}^{p} (1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \cdot \dfrac{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} - (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}{(1+\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}} + (1-\gamma_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n} T_i}}}}, \right.$$

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$$\bigcup_{\substack{\tau_1\in g_1,\\ \tau_2\in g_2,\\ \ldots,\tau_p\in g_p,\\ \tau_{p+1}\in g_{p+1}}} \frac{\dfrac{2\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\cdot\dfrac{2\tau_{p+1}^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}+(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}}{1+\left(1-\dfrac{2\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)\left(1-\dfrac{2(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}{(2-\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}+(\tau_{p+1})^{\frac{T_{p+1}}{\sum_{i=1}^{n}T_i}}}\right)}\Bigg)$$

$$=\left(\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\\ \gamma_p\in h_p,\gamma_{p+1}\in h_{p+1}}} \frac{\prod_{i=1}^{p+1}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{p+1}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p+1}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p+1}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.$$

$$\left.\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\ldots,\\ \tau_p\in g_p,\tau_{p+1}\in g_{p+1}}} \frac{2\prod_{i=1}^{p+1}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{p+1}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{p+1}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)$$

$$=\oplus_{\varepsilon}{}_{i=1}^{p+1}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)=\mathrm{DHFPEWA}(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_{p+1}).$$

Hence the theorem is proved for $p+1$ and thus true for all n.
Hence DHFPEWA $(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)$ is a DHFE.
This completes the proof of the theorem.

Theorem 2 (Idempotency) *Let $\widetilde{\alpha}_i=(h_i,g_i)$ $(i=1,2,\ldots,n)$ be a collections of* DHFE*s. If all $\widetilde{\alpha}_i$ $(i=1,2,\ldots,n)$ are equal, i.e., $\widetilde{\alpha}_i=\widetilde{\alpha}$ for all i, where $\widetilde{\alpha}=(h,g)$ then*

$$DHFPEWA\,(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\widetilde{\alpha}\,. \tag{10}$$

Proof. We have

$$\mathrm{DHFPEWA}\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)=\oplus_{\varepsilon}{}_{i=1}^{n}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)$$

$$=\left(\bigcup_{\gamma\in h}\frac{\prod_{i=1}^{n}(1+\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}(1-\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(1-\gamma)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\bigcup_{\tau\in g}\frac{2\prod_{i=1}^{n}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)$$

$$=\left(\bigcup_{\gamma\in h}\frac{(1+\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}-(1-\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}{(1+\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}+(1-\gamma)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}},\right.$$

$$\left.\bigcup_{\tau\in g}\frac{2\tau^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}{(2-\tau)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}+(\tau)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}\right)$$

$$=(h,g)=\widetilde{\alpha}\,.$$

Hence the theorem is proved.

Theorem 3 (Boundary) *Let $\widetilde{\alpha}_i=(h_i,g_i)$ $(i=1,2,\ldots,n)$ be a collections of DHFEs, and let*

$$\gamma_*=\min\{\gamma\in h_i\mid i=1,2,\ldots,n\},\qquad \gamma^*=\max\{\gamma\in h_i\mid i=1,2,\ldots,n\},$$
$$\tau_*=\min\{\tau\in g_i\mid i=1,2,\ldots,n\},\qquad \tau^*=\max\{\tau\in g_i\mid i=1,2,\ldots,n\},$$
$$\widetilde{\alpha}^-=(\gamma_*,\tau^*),\qquad \widetilde{\alpha}^+=(\gamma^*,\tau_*).$$

then

$$\widetilde{\alpha}^-\leqslant DHFPEWA\left(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\right)\leqslant\widetilde{\alpha}^+. \tag{11}$$

Proof. We have

$$DHFPEWA(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)=\oplus_{\varepsilon}{}_{i=1}^{n}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\widetilde{\alpha}_i\right)$$

$$=\left(\bigcup_{\substack{\gamma_1\in h_1,\\ \gamma_2\in h_2,\\ \ldots,\\ \gamma_n\in h_n}}\frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\bigcup_{\substack{\tau_1\in g_1,\\ \tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n}}\frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right).$$

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By the definition of γ_*, γ^*, τ_*, τ^*

$$\gamma_* \leqslant \gamma_i \leqslant \gamma^* \quad \text{for all } i, \text{ then}$$

Thus $\dfrac{1-\gamma^*}{1+\gamma^*} \leqslant \dfrac{1-\gamma_i}{1+\gamma_i} \leqslant \dfrac{1-\gamma_*}{1+\gamma_*}$ for all i

$$\text{i.e., } \prod_{i=1}^{n}\left(\frac{1-\gamma^*}{1+\gamma^*}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{1-\gamma_*}{1+\gamma_*}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \quad \text{for all } i$$

$$\text{i.e., } \frac{1-\gamma^*}{1+\gamma^*} \leqslant \prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \frac{1-\gamma_*}{1+\gamma_*} \quad \text{for all } i$$

$$\text{i.e., } \frac{2}{1+\gamma^*} \leqslant 1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \frac{2}{1+\gamma_*} \quad \text{for all } i$$

$$\text{i.e., } \gamma_* \leqslant \frac{2}{1+\prod_{i=1}^{n}\left(\frac{1-\gamma_i}{1+\gamma_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}} - 1 \leqslant \gamma^* \quad \text{for all } i$$

$$\text{i.e., } \gamma_* \leqslant \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} - \prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} + \prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}} \leqslant \gamma^* \quad \text{for all } i. \tag{12}$$

Similarly,
Since $\tau_* \leqslant \tau_i \leqslant \tau^*$ and $2-\tau^* \leqslant 2-\tau_i \leqslant 2-\tau_*$ then

$$\text{i.e., } \prod_{i=1}^{n}\left(\frac{2-\tau^*}{\tau^*}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{2-\tau_i}{\tau_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{2-\tau_*}{\tau_*}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \quad \text{for all } i$$

$$\text{i.e., } \frac{2}{\tau^*} \leqslant \prod_{i=1}^{n}\left(\frac{2-\tau_i}{\tau_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} + 1 \leqslant \frac{2}{\tau_*} \quad \text{for all } i$$

$$\text{i.e., } \tau_* \leqslant \frac{2}{\prod_{i=1}^{n}\left(\frac{2-\tau_i}{\tau_i}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} + 1} \leqslant \tau^* \quad \text{for all } i$$

$$\text{i.e., } \tau_* \leqslant \frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} + \prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}} \leqslant \tau^* \quad \text{for all } i. \tag{13}$$

Then from inequalities (12) and (13), and using (6) we obtain

$$s\left(\widetilde{\alpha}^{-}\right) \leqslant s\left(DHFPEWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right)\right) \leqslant s\left(\widetilde{\alpha}^{+}\right).$$

Therefore from the comparative laws of DHFE, it is clear that

$$\widetilde{\alpha}^{-} \leqslant DHFPEWA\left(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\right) \leqslant \widetilde{\alpha}^{+}.$$

This completes the proof of the theorem.

Theorem 4 (Additivity) *Let $\widetilde{\alpha}_i = (h_i, g_i)$ $(i = 1, 2, \ldots, n)$ be a collections of DHFEs, and if $\widetilde{\alpha} = (h, g)$ be another DHFE, then*

$$DHFPEWA(\widetilde{\alpha}_1 \oplus_\varepsilon \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_\varepsilon \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_\varepsilon \widetilde{\alpha}) = DHFPEWA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) \oplus_\varepsilon \widetilde{\alpha}.$$

Proof. Based on the operational laws of DHFEs, we have

$$\widetilde{\alpha}_i \oplus_\varepsilon \widetilde{\alpha} = \left(\bigcup_{\gamma_i \in h_i, \gamma \in h} \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}, \bigcup_{\tau_i \in g_i, \tau \in g} \frac{\tau_i \tau}{1 + (1 - \tau_i)(1 - \tau)} \right).$$

According to theorem 1, we have

$$\begin{aligned}
&DHFPEWA\left(\widetilde{\alpha}_1 \oplus_\varepsilon \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_\varepsilon \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_\varepsilon \widetilde{\alpha}\right) \\
&= \left(\bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \ldots, \\ \gamma_n \in h_n, \gamma \in h}} \frac{\prod_{i=1}^{n} \left(1 + \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{j=1}^{n} \left(1 - \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{j=1}^{n} \left(1 + \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{j=1}^{n} \left(1 - \frac{\gamma_i + \gamma}{1 + \gamma_i \gamma}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right. \\
&\left. \bigcup_{\substack{\tau_1 \in g_1, \tau_2 \in g_2, \\ \ldots, \\ \tau_n \in g_n, \tau \in g}} \frac{2 \prod_{i=1}^{n} \left(\frac{\tau_i \tau}{1 + (1 - \tau_i)(1 - \tau)}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} \left(2 - \frac{\tau_i \tau}{1 + (1 - \tau_i)(1 - \tau)}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} \left(\frac{\tau_i \tau}{1 + (1 - \tau_i)(1 - \tau)}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}} \right) \\
&= \left(\bigcup_{\substack{\gamma_1 \in h_1, \gamma_2 \in h_2, \\ \ldots, \\ \gamma_n \in h_n, \gamma \in h}} \frac{\prod_{i=1}^{n} (1 + \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 + \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 - \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n} (1 + \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 + \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} (1 - \gamma)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}, \right.
\end{aligned}$$

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$$\left.\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n,\tau\in g}} \frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}(2-\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}(\tau)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)$$

$$=\left(\bigcup_{\substack{\gamma_1\in h_1,\gamma_2\in h_2,\\ \ldots,\gamma_n\in h_n,\gamma\in h}} \frac{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.$$

$$\left.\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n,\tau\in g}} \frac{\tau.2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(2-\tau)\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\tau\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right).$$

Again from the operational laws of DHFE

$$DHFPEWA(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n)\oplus_{\varepsilon}\widetilde{\alpha}$$

$$=\left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n} \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.$$

$$\left.\bigcup_{\tau_1\in g_1,\tau_2\in g_2,\ldots,\tau_n\in g_n} \frac{2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)\oplus_{\varepsilon}(h,g)$$

$$=\left(\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\ldots,\gamma_n\in h_n,\gamma\in h} \frac{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(1+\gamma)\prod_{i=1}^{n}(1+\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(1-\gamma)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}},\right.$$

$$\left.\bigcup_{\substack{\tau_1\in g_1,\tau_2\in g_2,\\ \ldots,\\ \tau_n\in g_n,\tau\in g}} \frac{\tau.2\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{(2-\tau)\prod_{i=1}^{n}(2-\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\tau\prod_{i=1}^{n}(\tau_i)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right).$$

Thus,

$$DHFPEWA(\widetilde{\alpha}_1 \oplus_\varepsilon \widetilde{\alpha}, \widetilde{\alpha}_2 \oplus_\varepsilon \widetilde{\alpha}, \ldots, \widetilde{\alpha}_n \oplus_\varepsilon \widetilde{\alpha}) = DHFPEWA(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) \oplus_\varepsilon \widetilde{\alpha}.$$

This completes the proof.

Theorem 5 *Let* $\widetilde{\alpha}_i = (h_i, g_i)$ *($i = 1,2,\ldots,n$) be a collections of DHFEs, then the aggregated value by using DHFPEWG operator is also a DHFE and* $DHFPEWG(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) = \otimes_{\varepsilon}{}_{i=1}^{n} (\widetilde{\alpha}_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}$ *and*

$$\begin{aligned} DHFPEWG(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n) &= \otimes_{\varepsilon}{}_{i=1}^{n} (\widetilde{\alpha}_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} \\ &= \left(\bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \ldots, \gamma_n \in h_n} \frac{2\prod_{i=1}^{n} (\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^{n} (2-\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod_{i=1}^{n} (\gamma_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}, \right. \\ &\quad \left. \bigcup_{\tau_1 \in g_1, \tau_2 \in g_2, \ldots, \tau_n \in g_n} \frac{\prod_{i=1}^{n} (1+\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} - \prod_{i=1}^{n} (1-\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^{n} (1+\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}} + \prod_{i=1}^{n} (1-\tau_i)^{\frac{T_i}{\sum_{i=1}^n T_i}}} \right), \end{aligned}$$

where $T_i = \prod_{k=1}^{i-1} S(\widetilde{\alpha}_k)$ *($i = 2,3,\ldots,n$),* $T_1 = 1$*, and* $S(\widetilde{\alpha}_k)$ *is the score value of* DHFE$\widetilde{\alpha}_k$.

Proof. The proof of this theorem is similar to the proof of Theorem 1.

5. An approach to solve MCDM problems with DHFEs

Let $X = \{x_1, x_2, \ldots, x_m\}$ be the set of alternatives and let $C = \{c_1, c_2, \ldots, c_n\}$ be a collection of criteria and there prioritization is given as $c_1 \succ c_2 \succ \ldots \succ c_n$ in such a manner that criteria c_j has a higher priority than c_i, if $j < i$. Now $E = \{e_1, e_2, \ldots, e_p\}$ represents a set of decision makers and the linear ordering $e_1 \succ e_2 \succ e_3 \succ \ldots \succ e_p$ represents prioritization between the decision makers in such a manner that decision maker e_η has a higher priority than decision maker e_ξ if $\eta < \xi$. Suppose that the decision matrix $R^{(q)} = \left(\widetilde{r}_{ij}^{(q)}\right)_{m \times n}$ $(q = 1,2,\ldots,p)$ is in the form of dual hesitant fuzzy matrix. The elements of this matrix are represented by DHFEs as $\tilde{r}_{ij}^{(q)} = \left(h_{ij}^{(q)},\, g_{ij}^{(q)}\right)$ which designates the value of the alternative $x_i \in X$ on the criteria $c_j \in C$ provided by the decision maker e_q, where $h_{ij}^{(q)}$ designates the membership degree of the alternative x_i satisfies the criteria C_j expressed by the decision maker e_q; where as $g_{ij}^{(q)}$ indicates the non-membership degree of the same alternative corresponding to the same criteria.

Now utilizing the DHFPEWA and DHFPEWG operators to develop an approach to multi-criteria group decision making under dual hesitant fuzzy environment, the main steps are described as follows:

Step 1. Calculate the value of $T_{ij}^{(q)}(q = 1, 2, \ldots, p)$with the following equations.

$$T_{ij}^{(q)} = \prod_{k=1}^{q-1} S\left(\tilde{r}_{ij}^{(k)}\right) \qquad (q = 1, 2, \ldots, p), \tag{14}$$

$$T_{ij}^{(1)} = 1. \tag{15}$$

Step 2. To aggregate all the individual dual hesitant fuzzy decision matrix

$$R^{(q)} = \left(\tilde{r}_{ij}^{(q)}\right)_{m\times n} \qquad (q = 1, 2, \ldots, p).$$

Thus using the DHFPEWA operator

$$\tilde{r}_{ij} = \text{DHFPEWA}\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \ldots, \tilde{r}_{ij}^{(p)}\right)$$
$$= \left(\bigcup_{\gamma_{ij}^{(q)} \in h_{ij}^{(q)}} \frac{\prod_{q=1}^{p}\left(1+\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} - \prod_{q=1}^{p}\left(1-\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(1+\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(1-\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}, \right.$$
$$\left. \bigcup_{\tau_{ij}^{(q)} \in g_{ij}^{(q)}} \frac{2\prod_{q=1}^{p}\left(\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(2-\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}} \right) \tag{16}$$

or using the DHFPEWG operator

$$\widetilde{r}_{ij} = \text{DHFPEWG}\left(\widetilde{r}_{ij}^{(1)}, \widetilde{r}_{ij}^{(2)}, \ldots, \widetilde{r}_{ij}^{(p)}\right)$$

$$= \left(\bigcup_{\gamma_{ij}^{(q)} \in h_{ij}^{(q)}} \frac{2\prod_{i=1}^{p}\left(\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(2-\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(\gamma_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}, \right.$$

$$\left. \bigcup_{\tau_{ij}^{(q)} \in g_{ij}^{(q)}} \frac{\prod_{q=1}^{p}\left(1+\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} - \prod_{q=1}^{p}\left(1-\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}}{\prod_{q=1}^{p}\left(1+\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}} + \prod_{q=1}^{p}\left(1-\tau_{ij}^{(q)}\right)^{\frac{T_{ij}^{(q)}}{\Sigma_{q=1}^{p} T_{ij}^{(q)}}}} \right). \tag{17}$$

Step 3. Calculate the values of T_{ij} as follows:

$$T_{ij} = \prod_{k=1}^{j-1} S\left(\widetilde{r}_{ik}\right), \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n); \tag{18}$$

$$T_{i1} = 1, \qquad i = 1, 2, \ldots, m. \tag{19}$$

Step 4. Aggregate the DHFEs $\widetilde{r}_{ij}$ for each alternative x_i using the DHFPEWA (or DHFPEWG) operator as follows:

$$\widetilde{r}_{i} = \text{DHFPEWA}\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}\right)$$

$$= \left(\bigcup_{\gamma_{ij} \in h_{ij}} \frac{\prod_{j=1}^{n}\left(1+\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}} - \prod_{j=1}^{n}\left(1-\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(1+\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n}\left(1-\gamma_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}}, \right.$$

$$\left. \bigcup_{\tau_{ij} \in g_{ij}} \frac{2\prod_{j=1}^{n}\left(\tau_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}}{\prod_{j=1}^{n}\left(2-\tau_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}} + \prod_{j=1}^{n}\left(\tau_{ij}\right)^{\frac{T_{ij}}{\Sigma_{j=1}^{n} T_{ij}}}} \right) \tag{20}$$

or

$$\widetilde{r}_i = \text{DHFPEWG}\left(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}\right)$$

$$= \left(\bigcup_{\gamma_{ij}\in h_{ij}} \frac{2\prod_{j=1}^{n}\left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod_{j=1}^{n}\left(2-\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} + \prod_{j=1}^{n}\left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}, \right.$$

$$\left. \bigcup_{\tau_{ij}\in g_{ij}} \frac{\prod_{j=1}^{n}\left(1+\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} - \prod_{j=1}^{n}\left(1-\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod_{j=1}^{n}\left(1+\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}} + \prod_{j=1}^{n}\left(1-\tau_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}} \right). \tag{21}$$

Step 5. Rank all the alternatives by the proposed score function $S(\widetilde{r}_i)$ described in the above, then the highest value of $S(\widetilde{r}_i)$, the larger the overall $\widetilde{r}_i$, and thus the best alternative x_i, is determined.

Based on the methodology developed in this paper, the following illustrative example is considered and solved.

6. An illustrative example

To illustrate the efficiency of the developed *DHFPEW* operators a practical example, studied earlier by Yu [27] in intuitionistic fuzzy context, is adopted in dual hesitant fuzzy environment. The problem is then solved using the ranking process developed in this article and is compared with the process developed by Yu [27] and Yu et al. [30].

The problem under consideration is presented in summarised form as follows:

For enriching academic environment of a Chinese university, three decision makers viz., e_1, e_2 and e_3 in order of priority levels $e_1 > e_2 > e_3$, wants to appoint outstanding teachers among five candidates, x_i $(i = 1,\ 2,\ \ldots,\ 5)$ based on four criteria C_1, C_2, C_3, C_4. The criteria possesses the prioritization relationship as $C_1 > C_2 > C_3 > C_4$. After evaluating the five candidates with respect to their criteria, the decision makers constructed the following three decision matrices $R^{(n)} = (r_{ij}^{(n)})_{5\times 4}$ $(n = 1,\ 2, 3)$ using DHFEs as follows:

$$R^{(1)} = \begin{bmatrix} \langle\{0.55,0.9\},\{0.0,0.1\}\rangle & \langle\{0.6\},\{0.3\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle \\ \langle\{0.7,0.9\},\{0.0,0.1\}\rangle & \langle\{0.75\},\{0.15,0.25\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle \\ \langle\{0.9\},\{0.0\}\rangle & \langle\{0.75,0.85\},\{0.0,0.15\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.45\},\{0.1,0.45\}\rangle \\ \langle\{0.5,0.75\},\{0.15\}\rangle & \langle\{0.6,0.75\},\{0.15,0.25\}\rangle & \langle\{0.7,0.9\},\{0.0,0.1\}\rangle & \langle\{0.3\},\{0.6\}\rangle \\ \langle\{0.4,0.75\},\{0.15,0.25\}\rangle & \langle\{0.4,0.6\},\{0.3,0.4\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.5,0.6\},\{0.3\}\rangle \end{bmatrix},$$

$$R^{(2)} = \begin{bmatrix} \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.3,0.4\},\{0.6\}\rangle \\ \langle\{0.85,0.75\},\{0.05,0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.75,\},\{0.15\}\rangle \\ \langle\{0.7,0.9\},\{0.0,0.05\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.75\},\{0.05,0.15\}\rangle & \langle\{0.6,0.7\},\{0.1,0.3\}\rangle \\ \langle\{0.3,0.9\},\{0.0,0.1\}\rangle & \langle\{0.3,0.4\},\{0.6\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.6\},\{0.3\}\rangle \\ \langle\{0.45\},\{0.45,0.55\}\rangle & \langle\{0.6\},\{0.3\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.7,0.9\},\{0.0,0.1\}\rangle \end{bmatrix},$$

$$R^{(3)} = \begin{bmatrix} \langle\{0.75\},\{0.15,0.25\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.65,0.75\},\{0.15\}\rangle & \langle\{0.3\},\{0.4,0.6\}\rangle \\ \langle\{0.6\},\{0.1,0.3\}\rangle & \langle\{0.75,0.85\},\{0.15\}\rangle & \langle\{0.9\},\{0.0,0.1\}\rangle & \langle\{0.6\},\{0.3\}\rangle \\ \langle\{0.9\},\{0.0\}\rangle & \langle\{0.6\},\{0.3\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.8,0.9\},\{0.0,0.1\}\rangle \\ \langle\{0.5,0.9\},\{0.0,0.1\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15,0.25\}\rangle \\ \langle\{0.75\},\{0.15\}\rangle & \langle\{0.75\},\{0.15\}\rangle & \langle\{0.7,0.9\},\{0.0,0.1\}\rangle & \langle\{0.45\},\{0.45,0.55\}\rangle \end{bmatrix}.$$

To select the most preferable candidate the developed process is applied on the above matrices and the following steps are performed.

It is worthy to mention here that Step 1 is common for both the DHFPEWA and DHFPEWG operators.

Step 1. Calculate the value of $T_{ij}^{(i)}$ $(i = 1, 2, 3)$ using equations (16) and (17).

$$T_{ij}^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad T_{ij}^{(2)} = \begin{bmatrix} 0.8375 & 0.65 & 0.825 & 0.925 \\ 0.875 & 0.775 & 0.8 & 0.8 \\ 0.95 & 0.8625 & 0.825 & 0.5875 \\ 0.7375 & 0.7375 & 0.875 & 0.35 \\ 0.6875 & 0.575 & 0.8 & 0.625 \end{bmatrix},$$

$$T_{ij}^{(3)} = \begin{bmatrix} 0.6909 & 0.52 & 0.7631 & 0.3469 \\ 0.7438 & 0.7169 & 0.66 & 0.64 \\ 0.8431 & 0.7978 & 0.6806 & 0.4259 \\ 0.5716 & 0.2766 & 0.7219 & 0.2275 \\ 0.3266 & 0.3738 & 0.74 & 0.5469 \end{bmatrix}.$$

Step 2. Aggregate the three given decision matrices $R^{(k)}$ $(k = 1,2,3)$ by using DHFPEWA operator to aggregate the overall decision matrix R which is shown below:

$$R = \begin{bmatrix}
\left\langle \begin{Bmatrix} 0.6819, 0.7290, \\ 0.8244, 0.8522 \end{Bmatrix}, \begin{Bmatrix} 0.0, 0.1279, \\ 0.0, 0.1478 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7459 \end{Bmatrix}, \begin{Bmatrix} 0.0, \\ 0.1894 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7909, 0.8117, \\ 0.8285, 0.8459 \end{Bmatrix}, \begin{Bmatrix} 0.0, \\ 0.1320 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.6758, \\ 0.7002 \end{Bmatrix}, \begin{Bmatrix} 0.0, 0.0, 0.2711, \\ 0.2910 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.6927, 0.7388, \\ 0.7947, 0.8271 \end{Bmatrix}, \begin{Bmatrix} 0, 0.0795, 0, 0.1103, \\ 0, 0.1146, 0, 0.1579 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.8104 \\ 0.8366 \end{Bmatrix}, \begin{Bmatrix} 0.0, 0.1324, \\ 0.0, 0.1634 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.8030, \\ 0.8333 \end{Bmatrix}, \begin{Bmatrix} 0.0, \\ 0.1347 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7161 \end{Bmatrix}, \begin{Bmatrix} 0.1809 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.8528, \\ 0.9000 \end{Bmatrix}, \begin{Bmatrix} 0.0, \\ 0.0 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7822, \\ 0.8202 \end{Bmatrix}, \begin{Bmatrix} 0, 0, 0, \\ 0.1634 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7500, \\ 0.7954 \end{Bmatrix}, \begin{Bmatrix} 0.1052, \\ 0.1500 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5885, 0.6378, \\ 0.6207, 0.6670 \end{Bmatrix}, \begin{Bmatrix} 0, 0.1000, 0, 0.1394, \\ 0, 0.2189, 0, 0.2978 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.4404, 0.6051, 0.6880, \\ 0.7904, 0.5758, 0.7087, \\ 0.7729, 0.8500 \end{Bmatrix}, \{0, 0, 0, 0.1194\} \right\rangle & \left\langle \begin{Bmatrix} 0.5307, 0.5600, \\ 0.6231, 0.6480 \end{Bmatrix}, \begin{Bmatrix} 0.2596, \\ 0.3300 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7317, 0.7730, \\ 0.8227, 0.8512 \end{Bmatrix}, \begin{Bmatrix} 0.0, \\ 0.1285 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.4545 \end{Bmatrix}, \begin{Bmatrix} 0.4336, \\ 0.4620 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.4881, \\ 0.6675 \end{Bmatrix}, \begin{Bmatrix} 0.2227, 0.2414, \\ 0.2848, 0.3078 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5431, \\ 0.6332 \end{Bmatrix}, \begin{Bmatrix} 0.2639, \\ 0.3076 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.8003, \\ 0.8553 \end{Bmatrix}, \begin{Bmatrix} 0, 0, 0, \\ 0.1175 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5543, 0.6633, \\ 0.5985, 0.6987 \end{Bmatrix}, \begin{Bmatrix} 0, 0, 0.2475, \\ 0.2625 \end{Bmatrix} \right\rangle
\end{bmatrix}.$$

Step 3. To calculate the value of T_{ij} use the equation (20) and (21).

$$T_{ij} = \begin{bmatrix} 1 & 0.8515 & 0.7030 & 0.6162 \\ 1 & 0.8528 & 0.7460 & 0.6530 \\ 1 & 0.9382 & 0.8258 & 0.6793 \\ 1 & 0.8245 & 0.5341 & 0.4621 \\ 1 & 0.6568 & 0.4277 & 0.3846 \end{bmatrix}.$$

Step 4. Utilize DHFPEWA operator to aggregate all DHFEs $\widetilde{r}_{ij}$ ($i = 1$, 2, 3, 4, 5; $j = 1$, 2, 3, 4) for each alternative x_i to reduce it in DHFE $\widetilde{r}_i$ ($i = 1$, 2, 3, 4, 5).

Step 5. By the definition 3, calculate the score values $S(r_i)$ ($i = 1$, 2, 3, 4, 5) of the alternative x_i . The values are as follows:

$$S(r_1) = 0.8770, S(r_2) = 0.8846, S(r_3) = 0.9223, S(r_4) = 0.8162, S(r_5) = 0.8091.$$

Since $S_3 > S_2 > S_1 > S_4 > S_5$, the ordering of alternatives are found as

$$x_3 > x_2 > x_1 > x_4 > x_5 \,.$$

Now, the given problem is solved using DHFPEWG operator, for finding the preference ordering of the candidates. The following steps are performed:

Step 1. Same as above step 1.

Step 2. Utilize the DHFPEWG operator to aggregate the given dual hesitant fuzzy decision matrix $R^{(q)} = \left(\tilde{r}_{ij}^{(q)}\right)_{5\times 4}$ $(q = 1, 2, 3)$

$$R =$$

$$\begin{bmatrix}
\left\langle \begin{Bmatrix} 0.6671, 0.6981, \\ 0.8083, 0.8418 \end{Bmatrix}, \begin{Bmatrix} 0.0911, 0.1193, \\ 0.1303, 0.1582 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7126 \end{Bmatrix}, \begin{Bmatrix} 0.1857, \\ 0.2088 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7660, 0.7969, \\ 0.8044, 0.8358 \end{Bmatrix}, \begin{Bmatrix} 0.1026, \\ 0.1341 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5148, \\ 0.5709 \end{Bmatrix}, \begin{Bmatrix} 0.3336, 0.3697, \\ 0.3723, 0.4072 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.6870, 0.7188, \\ 0.7607, 0.7939 \end{Bmatrix}, \begin{Bmatrix} 0.0452, 0.1043, 0.0788, \\ 0.1375, 0.0833, 0.1420, \\ 0.1168, 0.1749 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7957, \\ 0.8246 \end{Bmatrix}, \begin{Bmatrix} 0.1038, 0.1345, \\ 0.1450, 0.1754 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7894, \\ 0.8220 \end{Bmatrix}, \begin{Bmatrix} 0.1101, \\ 0.1366 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7091 \end{Bmatrix}, \begin{Bmatrix} 0.1903 \end{Bmatrix} \right\rangle \\
\left\langle \{0.8302, 0.9000\}, \{0, 0.0170\} \right\rangle & \left\langle \begin{Bmatrix} 0.7498, \\ 0.7869 \end{Bmatrix}, \begin{Bmatrix} 0.0926, 0.1247, \\ 0.1485, 0.1802 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7500, \\ 0.7894 \end{Bmatrix}, \begin{Bmatrix} 0.1173, \\ 0.1500 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5589, 0.5765, \\ 0.5861, 0.6042 \end{Bmatrix}, \begin{Bmatrix} .0789, .1, .1392, , .2636, \\ .16, .2833, .3195, .3384 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.4277, 0.5072, 0.615, \\ 0.7135, 0.5195, 0.6094, \\ 0.7284, 0.8341 \end{Bmatrix}, \begin{Bmatrix} 0.0654, 0.09, \\ 0.0972, 0.1217 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.4898, 0.5385, \\ 0.5542, 0.6067 \end{Bmatrix}, \begin{Bmatrix} 0.3360, \\ 0.3811 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7306, 0.7635, \\ 0.8068, 0.8408 \end{Bmatrix}, \begin{Bmatrix} 0.0927, \\ 0.1308 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.4081 \end{Bmatrix}, \begin{Bmatrix} 0.4853, \\ 0.4967 \end{Bmatrix} \right\rangle \\
\left\langle \begin{Bmatrix} 0.4653, \\ 0.6379 \end{Bmatrix}, \begin{Bmatrix} 0.2590, 0.3010, \\ 0.3066, 0.3473 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5149, \\ 0.6275 \end{Bmatrix}, \begin{Bmatrix} 0.2721, \\ 0.3254 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.7813, \\ 0.8400 \end{Bmatrix}, \begin{Bmatrix} 0.0594, 0.0885, \\ 0.0909, 0.1198 \end{Bmatrix} \right\rangle & \left\langle \begin{Bmatrix} 0.5397, 0.5884, \\ 0.5865, 0.6376 \end{Bmatrix}, \begin{Bmatrix} 0.2586, 0.2897, \\ 0.2853, 0.3159 \end{Bmatrix} \right\rangle
\end{bmatrix}.$$

Step 3. Calculate the value of T_{ij} $(i = 1, 2, 3, 4, 5)$, $(j = 1, 2, 3, 4)$

$$T_{ij} = \begin{bmatrix} 1 & 0.8146 & 0.6172 & 0.5192 \\ 1 & 0.8149 & 0.6806 & 0.5725 \\ 1 & 0.9283 & 0.7574 & 0.6196 \\ 1 & 0.7629 & 0.4534 & 0.3794 \\ 1 & 0.6241 & 0.3971 & 0.3417 \end{bmatrix}.$$

Step 4. Utilize the DHFPEWG operator to aggregate all DHFEs $\tilde{r}_{ij}$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4)$ for each alternative x_i to reduce in DHFE $\tilde{r}_i$ $(i = 1, 2, 3, 4, 5)$.

Step 5. By Definition 3, calculate the score values $S(r_i)$ $(i = 1, 2, 3, 4, 5)$ of the alternative x_i. The score values are found as

$$S(r_1) = 0.7739, S(r_2) = 0.8154, S(r_3) = 0.8240, S(r_4) = 0.6742, S(r_5) = 0.6479.$$

Since $S_3 > S_2 > S_1 > S_4 > S_5$ the ordering is found as

$$x_3 > x_2 > x_1 > x_4 > x_5 \, .$$

It is evident that the ordering of the candidates are the same for both the operators.

Now, if the problem is considered in a hesitant fuzzy environment and is solved using hesitant fuzzy prioritized Einstein weighted averaging operator developed by Yu et al. [30] the score value of the candidates are found as

$$S(r_1) = 0.7673, S(r_2) = 0.7879, S(r_3) = 0.8009, S(r_4) = 0.6570, S(r_5) = 0.6367$$

with the ordering $x_3 > x_2 > x_1 > x_4 > x_5$.

But, if the problem is solved using hesitant fuzzy prioritized Einstein weighted geometric operator developed by Yu et al. [30] the score value of the candidates changed and are found as

$$S(r_1) = 0.7185, S(r_2) = 0.7681, S(r_3) = 0.7623, S(r_4) = 0.5896, S(r_5) = 0.5922$$

with the ordering $x_2 > x_3 > x_1 > x_5 > x_4$.

So, the methods developed by Yu et al. [30] are not found consistent in this context.

Further, if the problem under consideration is solved in intuitionistic fuzzy environment using the technique developed by Yu [27], the same inconsistencies are observed as in the case of Yu et al. [30]. In this context the solutions are found as

$$S(r_1) = 0.8901, S(r_2) = 0.8940, S(r_3) = 0.9003, S(r_4) = 0.8737, S(r_5) = 0.8574$$

using intuitionistic fuzzy prioritized averaging operator with the rank of the alternatives

$$x_3 > x_2 > x_1 > x_4 > x_5$$

and using intuitionistic fuzzy prioritized geometric operator

$$S(r_1) = 0.7586, S(r_2) = 0.8127, S(r_3) = 0.7983, S(r_4) = 0.6956, S(r_5) = 0.7097$$

with the rank

$$x_2 > x_3 > x_1 > x_5 > x_4 \, .$$

Thus the proposed method is consistent than the previous approaches and provides efficient solutions in the decision making context.

7. Conclusions

Most of the traditional hesitant fuzzy aggression operators are based on algebraic operations. However, algebraic sum and algebraic product are not only the operations for aggregation of HFS. Many aggression operators are available to solve group decision making problems in DHF environment. But those operators did not provide consistent satisfactory solution in the decision making environment. In this paper DHFPEWA and DHFPEWG are proposed which establishes their capabilities to provide efficient solution in the decision making process. A new score function for DHFEs is proposed to remove the drawback of earlier methods [28]. It is also to be noted here that this process evaluates the weights of the decision makers from the decision matrix not by assigning arbitrary weights to them. Thus the influence of outside values cannot affect the decision of the proposed model. The proposed method can be extended to solve MCDM problems in interval valued DHF as well as dual hesitant probabilistic fuzzy environment without any computational complexities. However, it is hoped that the developed method can add an extra dimension in the process of making decision in hesitant fuzzy contexts.

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